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Three term recurrence relation modulo ideal and orthogonality of polynomials of several variables

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Abstract

Orthogonality of polynomials in several variables with respect to a positive Borel measure supported on an algebraic set is the main theme of this paper. As a step towards this goal quasi-orthogonality with respect to a non-zero Hermitian linear *functional* is studied in detail; this occupies a substantial part of the paper. Therefore necessary and sufficient conditions for quasi-orthogonality in terms of the three term recurrence relation modulo a polynomial ideal are accompanied with a thorough discussion. All this enables us to consider orthogonality in full generality. Consequently, a class of simple objects missing so far, like spheres, is included. This makes it important to search for results on existence of *measures* representing orthogonality on *algebraic sets*; a general approach to this problem fills up the three final sections.

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0. Introduction

Orthogonal polynomials constitute a vital part of Analysis continuously penetrating other areas of Mathematics and much beyond. One of the basic questions of their theory is to decide

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whether a sequence of polynomials is orthogonal. In the case of a *single* real variable the most powerful characterization is given by the famous three term recurrence relation. More precisely, orthogonality of a sequence of polynomials with respect to a linear functional reduces immediately the linear dependence of multiplication by the independent variable on these polynomials to a three term relation of recurrence type. The other way, that is from the three term recurrence relation to existence of a linear functional orthogonalizing the sequence of polynomials is established by a result which is commonly known as Favard's Theorem. The important feature of the single variable case is that the linear functional, if positive, is always determined by a measure.

In the *several* variable case the situation is much more complex. Even the meaning of the three term recurrence relation is dubious. The very first difficulty is in finding convenient notation (related to the order in which the orthonormalization procedure has to be performed) which allows us to see the recurrence relation as a three term one. The pioneering attempt in this direction was made by Kowalski [19,20]. A decade later the theme was undertaken by Xu [30–32] and independently by Gekhtman and Kalyuzhny [14,15] (see also [33,34] for further investigations along these lines and [11] for a recent account of the theory).

Further difference is in the fact that the three term recurrence relation may not determine any orthogonality measure though the functional orthogonality in Favard's Theorem is still preserved; regardless the way the relation is built up.

The three term recurrence relation considered in the references alluded to so far forces the Zariski closure of the support of an orthogonalizing measure, provided it exists, to be the whole space $\mathbb{R}^N$ (in fact, this is the only essential case in a single variable). However, important instances, like a sphere, are left out of the game (cf. [11, p. 126]) which calls for extending the study to cases of measures not having too massive support. Our work is intended to get rid of this incompleteness introducing recurrence relations of matrix type satisfied modulo an ideal.

The principal observation is that, if any orthogonality measure exists then the aforesaid ideal consists of all polynomials vanishing on the Zariski closure of its support. The main task is to go the other way around: given an ideal, find (necessary and) sufficient conditions for it to admit measures representing orthogonality of a sequence of polynomials in question. A class of ideals we distinguish for that, called *ideals of type* C, has the property that three term recurrence relations modulo an ideal it induces automatically imply the existence of orthogonalizing measures. However it is not easy to find proper tools to work on this class. Fortunately, the class of ideals of type C contains the cases of algebraic sets of type A and B considered in [24], which can be handled by means of functional analysis and operator theory. This allows us to work out further properties of types A and B, and consequently of type C as well. As a result we get that ideals composed of polynomials vanishing on a compact algebraic set are of type C; other classes considered in this paper correspond to some unbounded algebraic sets. Then basing on a result of [24] we indicate an implicit example of a non-zero proper ideal which is not of type C (we do not know if the zero ideal is of type C). In the case of such ideals we impose some extra conditions (relying on the well know operator theory result of Nelson [21]) on the matrix coefficients appearing in the three term recurrence relation so as to ensure the existence of orthogonalizing measures.

Implementing the programme already described we devote a substantial part of the paper to the so-called *quasi-orthogonality*. It turns out that this notion, due to its simple algebraic nature, allows us to extract the most important properties (like those of Section 6, for instance) of the three term recurrence relation, resulting in a number of versions of Favard's theorem. According to this arrangement the orthogonality with respect to a positive functional follows; one of the key arguments in making it possible is to find an appropriate basis of monomials in the quotient space $\mathbb{C}[X_1, \dots, X_N]/V$, V an ideal, playing the role of the standard basis of monomials in $\mathbb{C}[X_1, \dots, X_N]$. Needless to say that after taking $V = \{0\}$ our considerations cover those of [11].

Let us give a short summary of the paper. The initial four sections contain basic ingredients including the notion of a rigid *V*-basis, where *V* is a polynomial ideal. In the subsequent section we formulate a variety of results on quasi-orthogonality of polynomials of several variables with respect to a Hermitian linear functional. This culminates in Theorem 18 which is a far-reaching generalization of the classical Favard theorem. The miscellaneous features of Theorem 18 are discussed in Sections 6 and 7. In particular, Proposition 26 shows that the rank condition may be replaced by the requirement on degrees of involved polynomials, Corollary 28 is a refinement of a result of [31], while Theorem 30 is a "complex" version of Theorem 18. Orthogonality of polynomials of several variables with respect to a positive definite linear functional is investigated in Section 8, with Theorem 36 as the main result. Proposition 41 in Section 9 shows that an orthogonalizing functional *L* comes from a positive Borel measure on $\mathbb{R}^N$ only if the attached ideal $\mathcal{V}_L$ (see (43)) is a set ideal, which leads directly to algebraic sets and the Zariski topology. Finally, the last three sections deal with the question of existence of orthogonalizing measures. This can be affirmed by numerous criteria, either in terms of set ideals (e.g. Theorem 43) or in terms of the matrix coefficients in the three term recurrence relation (e.g. Theorem 56). Furthermore some open questions are raised in Sections 10 and 11.

1. Prerequisites

Denote by card *A* the cardinality of a set *A*. Put $\mathbb{N} = \{0, 1, \dots\}$ and

$$\overline{i, j} = \{k \in \mathbb{N} : i \leqslant k \leqslant j\} \quad \text{for } i \in \mathbb{N} \quad \text{and} \quad j \in \mathbb{N} \cup \{\infty\}.$$

As usual $\mathbb{R}$ (resp. $\mathbb{C}$) stands for the field of all real (resp. complex) numbers, and $\delta_{i,j}$ for the Kronecker symbol. Write lin *A* for the linear span of a subset *A* of a linear space. Denote by $\mathbb{N}^N$ the *N*-fold Cartesian product of $\mathbb{N}$ by itself. Set $|\alpha| = \alpha_1 + \cdots + \alpha_N$ for $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$. Let $\mathcal{P}_N$ stand for the algebra of all polynomials in *N* commuting indeterminates $X_1, \dots, X_N$ with complex coefficients (if $N = 1$, we simply write *X* instead of X_1). Members of $\mathcal{P}_N$ are customarily identified with complex polynomial functions on $\mathbb{R}^N$. Equip the algebra $\mathcal{P}_N$ with the unique involution $p \mapsto p^*$ such that $X_i^* = X_i$ for all $i = 1, \dots, N$. Set $F^* = \{p^* : p \in F\}$ for $F \subseteq \mathcal{P}_N$. Notice that if $p \in \mathcal{P}_N$, then $p = p^*$ if and only if *p* is a real polynomial (i.e. all the coefficients of *p* are real). Put $\mathfrak{Re}\, p = \frac{1}{2}(p + p^*)$ and $\mathfrak{Im}\, p = \frac{1}{2\mathrm{i}}(p - p^*)$ for $p \in \mathcal{P}_N$. Write $\mathcal{R}_N$ for the set $\{\mathfrak{Re}\, p : p \in \mathcal{P}_N\}$ which is the ring of all real polynomials in *N* commuting indeterminates $X_1, \dots, X_N$. Set

$$\mathcal{P}_N^{\langle k \rangle} = \{p \in \mathcal{P}_N : \deg p \leqslant k\}, \quad k \in \mathbb{N},$$

$$\Lambda_N = \{X^\alpha : \alpha \in \mathbb{N}^N\},$$
$$\Lambda_N^{[k]} = \{X^\alpha : \alpha \in \mathbb{N}^N,\ |\alpha| = k\}, \quad k \in \mathbb{N},$$

where deg p stands for the degree of a polynomial p and $X^\alpha = X_1^{\alpha_1} \dots X_N^{\alpha_N}$ for all $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$. The ideal generated by a set $\{p_1, \dots, p_n\} \subseteq \mathcal{P}_N$ is denoted by $(p_1, \dots, p_n)$. According to the Hilbert basis theorem, every ideal in $\mathcal{P}_N$ is generated by a finite set of polynomials.

By a *matrix polynomial* (of size $m \times n$) we understand a polynomial with scalar matrix coefficients (of size $m \times n$). In particular, we can talk of row and column polynomials. Given a row (resp. column) polynomial Q, we denote by $\ell(Q)$ its *length* (i.e. $\ell(Q)$ is equal to the number of entries of Q). It is clear that a matrix polynomial P (of size $m \times n$) can be identified with a polynomial matrix $[p_{kl}]_{k=1}^{m}{}_{l=1}^{n}$, $p_{kl} \in \mathcal{P}_N$, and that under this identification we have

$$\deg P = \max\{\deg p_{kl} : k = 1, \dots, m,\ l = 1, \dots, n\}.$$

The matrix polynomial P is said to be *real* if all its entries p_{kl} are real polynomials. Given a matrix polynomial $P = \sum_{|\alpha| \leqslant n} A_\alpha X^\alpha$ with scalar matrix coefficients A_α, we set $P^* = \sum_{|\alpha| \leqslant n} A_\alpha^* X^\alpha$, where A_α^* is the adjoint of A_α. If P is written in a polynomial matrix form $P = [p_{kl}]_{k=1}^{m}{}_{l=1}^{n}$ with $p_{kl} \in \mathcal{P}_N$, then $P^* = [q_{kl}]_{k=1}^{n}{}_{l=1}^{m}$ with $q_{kl} = p_{lk}^*$; $P^\top$ stands for the transpose of P, i.e. $P^\top = [q_{kl}]_{k=1}^{n}{}_{l=1}^{m}$ with $q_{kl} = p_{lk}$. Given $p \in \mathcal{P}_N$ and a column polynomial $Q = [q_1, \dots, q_n]^\top$ with $q_j \in \mathcal{P}_N$, we set

$$pQ = [pq_1, \dots, pq_n]^\top.$$

A linear functional $L : \mathcal{P}_N \to \mathbb{C}$ is said to be *Hermitian* if $L(p^*) = \overline{L(p)}$ for all $p \in \mathcal{P}_N$. It is clear that a linear functional L on $\mathcal{P}_N$ is Hermitian if and only if $L(X^\alpha) \in \mathbb{R}$ for all $\alpha \in \mathbb{N}^N$. A linear functional L on $\mathcal{P}_N$ can also be considered as a mapping operating on matrix polynomials via

$$L([p_{kl}]_{k=1}^{m}{}_{l=1}^{n}) \stackrel{\text{df}}{=} [L(p_{kl})]_{k=1}^{m}{}_{l=1}^{n}, \quad p_{kl} \in \mathcal{P}_N.$$

For simplicity of notation we do not indicate the dependence of the so-defined mappings on the size of matrices. It is easily seen that if L is a Hermitian linear functional on $\mathcal{P}_N$, then for any matrix polynomial P

$$L(P^*) = L(P)^*.$$

Moreover, if L is a linear functional on $\mathcal{P}_N$, P is a matrix polynomial and A, B are scalar matrices for which the product APB makes sense, then

$$L(APB) = AL(P)B \quad \text{and} \quad L(P^\top) = L(P)^\top.$$

As usual, $\ker L$ stands for the kernel of a linear functional $L : \mathcal{P}_N \to \mathbb{C}$.

2. Monomial bases related to an ideal V

Let V be a proper ideal in $\mathcal{P}_N$. Denote by $\mathcal{P}_N/V$ the quotient algebra (i.e. $\mathcal{P}_N/V$ is the algebra of all cosets $p+V$, $p \in \mathcal{P}_N$) and by $\Pi_V : \mathcal{P}_N \longrightarrow \mathcal{P}_N/V$ the quotient mapping (i.e. $\Pi_V(p) \stackrel{\mathrm{df}}{=} p + V$, $p \in \mathcal{P}_N$). It will be convenient to extend the equality modulo the ideal V to matrix polynomials. Given two matrix polynomials $P = [p_{kl}]_{k=1}^{m}{}_{l=1}^{n}$ and $Q = [q_{kl}]_{k=1}^{m}{}_{l=1}^{n}$, we write $P \stackrel{V}{=} Q$ if $p_{kl} - q_{kl} \in V$ for all k, l. The following property of "$\stackrel{V}{=}$" will be used without explicit referring to:

> if $P \stackrel{V}{=} Q$, then $RPS \stackrel{V}{=} RQS$ for all matrix polynomials R and S for which the products RPS and RQS make sense.

Set

$$d_V(k) = \begin{cases} \dim \Pi_V\left(\mathcal{P}_N^{\langle 0]}\right) = 1 & \text{for } k = 0, \\ \dim \Pi_V\left(\mathcal{P}_N^{\langle k]}\right) - \dim \Pi_V\left(\mathcal{P}_N^{\langle k-1]}\right) & \text{for } k \geqslant 1 \end{cases}$$

and

$$\varkappa_V = \sup\{j \geqslant 0 : d_V(j) \neq 0\} \in \mathbb{N} \cup \{\infty\}.$$

Our first aim is to construct a particular (linear) basis of $\mathcal{P}_N/V$ composed of monomials. Let us fix a total order $\preceq$ on $\mathbb{N}^N$ satisfying the following condition: [1]

$$\text{if } \alpha, \beta \in \mathbb{N}^N \quad \text{and} \quad |\alpha| < |\beta|, \text{ then } \alpha \preceq \beta. \tag{1}$$

We write $\alpha \prec \beta$ in the case in which $\alpha \preceq \beta$ and $\alpha \neq \beta$. Using recursion, we define the sequence $\{\Sigma_k^V\}_{k=0}^{\infty}$ of subsets of Λ_N via:

$$\Sigma_0^V = \{X^0\} \quad \text{and}$$

$$\Sigma_{k+1}^V = \Bigg\{ X^\alpha \in \bigcup_{j=1}^{N} X_j \Sigma_k^V : \\ \Pi_V(X^\alpha) \notin \operatorname{lin} \Pi_V \left(\{X^0\} \cup \left\{ X^\beta \in \bigcup_{i=0}^{k} \bigcup_{j=1}^{N} X_j \Sigma_i^V : \beta \prec \alpha \right\} \right) \Bigg\},$$

where $X_j \Sigma_k^V \stackrel{\mathrm{df}}{=} \{X_j p : p \in \Sigma_k^V\}$. Set

$$\Lambda_N^V = \bigcup_{k=0}^{\infty} \Sigma_k^V.$$

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[1] Here is an example of a total order satisfying (1): $\alpha \preceq \beta$ if and only if either $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and α precedes β with respect to the lexicographic order.

It is clear that $\Sigma_k^V \subseteq \Lambda_N^{[k]}$ for all $k \geqslant 0$. However, it may happen that $\Sigma_k^V = \varnothing$ for some $k \geqslant 1$ (and consequently $\Sigma_j^V = \varnothing$ for all $j \geqslant k$); for example if $V = (X_1, \ldots, X_N) \subseteq \mathcal{P}_N$, then $\Sigma_1^V = \varnothing$.

Proposition 1. *If V is a proper ideal in $\mathcal{P}_N$, then for every $k \geqslant 0$*

(i) $\operatorname{card} \Lambda_N^V \cap \mathcal{P}_N^{\langle k\rangle} = \operatorname{card} \Pi_V\big(\Lambda_N^V \cap \mathcal{P}_N^{\langle k\rangle}\big)$,
(ii) *the set $\Pi_V\big(\Lambda_N^V \cap \mathcal{P}_N^{\langle k\rangle}\big)$ is a basis of $\Pi_V(\mathcal{P}_N^{\langle k\rangle})$,*
(iii) *$\Pi_V\big(\Lambda_N^V\big)$ is a basis of $\mathcal{P}_N/V$.*

Proof. Set $\Omega = \{\alpha \in \mathbb{N}^N : X^\alpha \in \Lambda_N^V\}$. Using induction, we show that for every $\alpha \in \Omega$, the following implication holds:

$$\text{if } \{a_\beta\}_{\substack{\beta\in\Omega \\ \beta\preceq\alpha}} \subseteq \mathbb{C} \quad \text{and} \quad \sum_{\substack{\beta\in\Omega \\ \beta\preceq\alpha}} a_\beta \Pi_V\big(X^\beta\big) = 0, \quad \text{then} \quad a_\beta = 0 \quad \text{for all } \beta\preceq\alpha. \tag{2}$$

Since $V \neq \mathcal{P}_N$, (2) is valid for $\alpha = 0$. Suppose that (2) holds for a fixed $\alpha \in \Omega$ and let $\gamma \in \Omega$ be the successor of α in Ω. We claim that γ satisfies (2). Suppose that, contrary to our claim, $a_\delta \neq 0$ for some $\delta \in \Omega$, $\delta\preceq\gamma$. We can assume, without loss of generality, that δ is the greatest element of Ω with this property. Since the case $\delta\preceq\alpha$ leads to a contradiction, we must have $\delta = \gamma$. Hence

$$\Pi_V(X^\gamma) = -\sum_{\substack{\beta\in\Omega \\ \beta\prec\gamma}} \frac{a_\beta}{a_\gamma}\, \Pi_V\big(X^\beta\big),$$

which contradicts the recursion definition of Λ_N^V. It follows directly from (2) that the set $\Pi_V(\Lambda_N^V)$ is linearly independent and that $\Pi_V|_{\Lambda_N^V}$ is injective. This implies (i) and a part of (ii). Since (iii) is an immediate consequence of (ii), all we have to prove is that $\Pi_V(\mathcal{P}_N^{\langle k\rangle}) = \operatorname{lin} \Pi_V\big(\Lambda_N^V \cap \mathcal{P}_N^{\langle k\rangle}\big)$. We do it by induction on k. The case $k = 0$ is obvious. Suppose it is true for a fixed $k \geqslant 0$. Take $\alpha \in \mathbb{N}^N$ with $|\alpha| = k + 1$. Then there are $j \in \{1, \ldots, N\}$ and $\beta \in \mathbb{N}^N$ such that $|\beta| = k$ and $X^\alpha = X_j X^\beta$. By the induction hypothesis, $\Pi_V(X^\beta) \in \operatorname{lin} \Pi_V\big(\bigcup_{i=0}^k \Sigma_i^V\big)$ and so $\Pi_V(X^\alpha) \in \operatorname{lin} \Pi_V\big(\bigcup_{i=0}^k X_j \Sigma_i^V\big)$. This shows that $\Pi_V(\Lambda_N^{[k+1]}) \subseteq \operatorname{lin} \Pi_V(\bigcup_{i=0}^k \bigcup_{j=1}^N X_j \Sigma_i^V)$. Thus, once more by the induction hypothesis, it is sufficient to prove that

$$\Pi_V(\widetilde{\Sigma}_{k+1}) \subseteq \operatorname{lin} \Pi_V \left(\bigcup_{i=0}^{k+1} \Sigma_i^V\right), \tag{3}$$

where $\widetilde{\Sigma}_{k+1} \stackrel{\text{df}}{=} \bigcup_{j=1}^N X_j \Sigma_k^V$. Suppose (3) is false. Then there is $\gamma \in \mathbb{N}^N$ such that

$$X^\gamma \in \widetilde{\Sigma}_{k+1} \quad \text{and} \quad \Pi_V(X^\gamma) \notin \operatorname{lin} \Pi_V \left(\bigcup_{i=0}^{k+1} \Sigma_i^V\right). \tag{4}$$

We can assume without loss of generality that γ is the least element of $\mathbb{N}^N$ satisfying (4). This yields:

$$\text{if } X^\beta \in \widetilde{\Sigma}_{k+1} \quad \text{and} \quad \beta \lessdot \gamma, \quad \text{then } \Pi_V(X^\beta) \in \operatorname{lin} \Pi_V\left(\bigcup_{i=0}^{k+1} \Sigma_i^V\right). \tag{5}$$

By (4), we must have $X^\gamma \notin \Sigma_{k+1}^V$. Thus, by the recurrence definition of Σ_{k+1}^V and the induction hypothesis, we have

$$\begin{aligned}\Pi_V(X^\gamma) \in \operatorname{lin} \Pi_V & \left(\Sigma_0^V \cup \left\{X^\beta \in \bigcup_{i=0}^{k} \bigcup_{j=1}^{N} X_j \Sigma_i^V : \beta \lessdot \gamma\right\}\right) \\ &= \operatorname{lin} \Pi_V\left(\Sigma_0^V \cup \bigcup_{i=0}^{k-1} \bigcup_{j=1}^{N} X_j \Sigma_i^V\right) + \operatorname{lin} \Pi_V\big(\{X^\beta \in \widetilde{\Sigma}_{k+1} : \beta \lessdot \gamma\}\big) \\ &\subseteq \operatorname{lin} \Pi_V\left(\bigcup_{i=0}^{k} \Sigma_i^V\right) + \operatorname{lin} \Pi_V\big(\{X^\beta \in \widetilde{\Sigma}_{k+1} : \beta \lessdot \gamma\}\big). \end{aligned} \tag{6}$$

According to (5), the second term in (6) is contained in $\operatorname{lin} \Pi_V\big(\bigcup_{i=0}^{k+1} \Sigma_i^V\big)$, which leads to $\Pi_V(X^\gamma) \in \operatorname{lin} \Pi_V(\bigcup_{i=0}^{k+1} \Sigma_i^V)$. This contradicts (4). $\square$

It follows from part (ii) of Proposition 1 that the sequence $\{\Sigma_k^V\}_{k=0}^\infty$ can also be defined by the following recursion: $\Sigma_0^V = \{X^0\}$ and

$$\Sigma_{k+1}^V = \left\{X^\alpha \in \bigcup_{j=1}^{N} X_j \Sigma_k^V : \right. \\ \left. \Pi_V(X^\alpha) \notin \operatorname{lin} \Pi_V\left(\mathcal{P}_N^{\langle k]} \cup \left\{X^\beta \in \bigcup_{j=1}^{N} X_j \Sigma_k^V : \beta \lessdot \alpha\right\}\right)\right\}.$$

Proposition 2. *If V is a proper ideal in $\mathcal{P}_N$, then for every $k \geqslant 0$,*

$$\operatorname{card} \Sigma_k^V = d_V(k), \tag{7}$$

$$d_V(k+j) \leqslant N^j d_V(k), \quad j \geqslant 0. \tag{8}$$

Proof. One can deduce from Proposition 1 (i) that the sets $\{\Pi_V\big(\Sigma_k^V\big)\}_{k=0}^\infty$ are pairwise disjoint and that $\operatorname{card} \Sigma_k^V = \operatorname{card} \Pi_V\big(\Sigma_k^V\big)$ for all $k \geqslant 0$. This and the condition (ii) of Proposition 1 imply $\operatorname{card} \Sigma_0^V = 1 = d_V(0)$ and

$$\begin{aligned}\operatorname{card} \Sigma_k^V &= \operatorname{card} \Pi_V\left(\bigcup_{i=0}^{k} \Sigma_i^V\right) \setminus \Pi_V\left(\bigcup_{i=0}^{k-1} \Sigma_i^V\right) \\ &= \dim \Pi_V\big(\mathcal{P}_N^{\langle k]}\big) - \dim \Pi_V\big(\mathcal{P}_N^{\langle k-1]}\big), \quad k \geqslant 1.\end{aligned}$$

Assertion (8) follows from inclusions $\Sigma_{k+1}^V \subseteq \bigcup_{j=1}^N X_j \Sigma_k^V$, $k \geqslant 0$. $\square$

3. *V*-bases

Let V be a proper ideal in $\mathcal{P}_N$, F be a linear subspace of $\mathcal{P}_N$ and B be a subset of $\mathcal{P}_N$. The set B is said to be *linearly V-independent*, if $\Pi_V(B)$ is a linearly independent subset of $\mathcal{P}_N/V$ and $\Pi_V|_B$ is injective. We say that F is a *linear V-span* of B, if $B \subseteq F$ and $\Pi_V(F) = \operatorname{lin} \Pi_V(B)$. Finally, B is said to be a (linear) *V-basis* of F, if B is linearly V-independent and F is a linear V-span of B. Clearly, B is a V-basis of F if and only if $B \subseteq F$, $\Pi_V(B)$ is a basis of $\Pi_V(F)$ and $\Pi_V|_B$ is injective. It is obvious that every V-basis of F is at most countable. We say that a sequence $\{Y_k\}_{k=0}^n$ $(0 \leqslant n \leqslant \infty)$ of column[2] polynomials is a *column representation* of a non-empty subset B of $\mathcal{P}_N$ if every element of B is an entry of exactly one column Y_i and for every $k \in \overline{0, n}$, entries of Y_k are pairwise distinct elements of B. It will be convenient to identify V-bases with their column representations. Namely, a sequence $\{Y_k\}_{k=0}^n$ of column polynomials is called a *V-basis* of F if $\{Y_k\}_{k=0}^n$ is a column representation of a V-basis of F. Likewise, we define the *linear V-independence* and the *linear V-span* of a sequence of column polynomials.

The proof of the following fact is left to the reader. Notice that (ii) implies that $\{Y_k\}_{k=0}^n$ is a column representation of a subset of $\mathcal{P}_N$. A sequence $\{A_k\}_{k=0}^n$ of (scalar) matrices is said to be *finite*, if $\operatorname{card}\{k : A_k \neq 0\} < \infty$.

Lemma 3. *Let V be a proper ideal in $\mathcal{P}_N$, F be a linear subspace of $\mathcal{P}_N$ and m be a positive integer. If $\{Y_k\}_{k=0}^n$ $(0 \leqslant n \leqslant \infty)$ is a sequence of column polynomials with entries in F, then the following conditions are equivalent*:

(i) *$\{Y_k\}_{k=0}^n$ is a V-basis of F (resp. $\{Y_k\}_{k=0}^n$ is linearly V-independent),*
(ii) *for every column polynomial P of length m with entries in F, there exists a unique (resp. at most one) finite sequence $\{A_k\}_{k=0}^n$ of scalar matrices with m rows such that $P \overset{V}{=} \sum_{k=0}^n A_k Y_k$.*

By Proposition 2, for every $k \in \overline{0, \varkappa_V}$, Σ_k^V is a non-empty subset of $\Lambda_N^{[k]}$. Depending on the context in which Σ_k^V appears, it is convenient to regard it either as a set or as a column polynomial (the same convention applies to other column sequences considered in this paper). In the latter case, entries of the column Σ_k^V are arranged in accordance with the lexicographical ordering (the length of the column Σ_k^V is equal to $d_V(k)$). By Propositions 1 and 2, $\{\Sigma_k^V\}_{k=0}^{\varkappa_V}$ is a V-basis of $\mathcal{P}_N$ which satisfies the following two conditions for every $k \in \overline{0, \varkappa_V}$:

$$\{\Sigma_i^V\}_{i=0}^k \text{ is a } V\text{-basis of } \mathcal{P}_N^{\langle k \rangle}, \tag{9}$$

$$\Sigma_{k+1}^V \subseteq \bigcup_{j=1}^N X_j \Sigma_k^V \quad (\text{here } \Sigma_k^V \text{ and } \Sigma_{k+1}^V \text{ are interpreted as sets}). \tag{10}$$

We shall show by means of $\{\Sigma_k^V\}_{k=0}^{\varkappa_V}$ that the numbers $d_V(k)$, $k \in \mathbb{N}$, determine the ideal V in a sense.

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[2] Columns are assumed to be finite and to have at least one entry.

Proposition 4. *If V_1 and V_2 are proper ideals in $\mathcal{P}_N$ such that $V_1 \subseteq V_2$ and $d_{V_1}(k) = d_{V_2}(k)$ for all $k \in \mathbb{N}$, then $V_1 = V_2$.*

Proof. Since $d_{V_1} = d_{V_2}$, we obtain $\varkappa_{V_1} = \varkappa_{V_2}$. Set $\varkappa \overset{\mathrm{df}}{=} \varkappa_{V_1}$.

We claim that for every $n \in \overline{0, \varkappa}$, $\{\Sigma_k^{V_2}\}_{k=0}^n$ is a V_1-basis of $\mathcal{P}_N^{\langle n \rangle}$. Since $V_1 \subseteq V_2$, one can deduce from Lemma 3 and (9) that $\{\Sigma_k^{V_2}\}_{k=0}^n$ is linearly V_1-independent. It follows from (7) that

$$\begin{aligned}\operatorname{card} \Sigma_0^{V_2} + \cdots + \operatorname{card} \Sigma_n^{V_2} &= d_{V_2}(0) + \cdots + d_{V_2}(n) = d_{V_1}(0) + \cdots + d_{V_1}(n)\\ &= \dim \Pi_{V_1}(\mathcal{P}_N^{\langle n \rangle}),\end{aligned}$$

which, together with $\deg \Sigma_k^{V_2} = k$, implies our claim.

Since $\{\Sigma_k^{V_1}\}_{k=0}^{\varkappa}$ is a V_1-basis of $\mathcal{P}_N$, we conclude that for every $p \in \mathcal{P}_N$, there exists $q \in \mathcal{P}_N$ such that $\deg q \leqslant \varkappa$ and $p \overset{V_1}{=} q$. This and the previous paragraph imply that $\{\Sigma_k^{V_2}\}_{k=0}^{\varkappa}$ is a V_1-basis of $\mathcal{P}_N$. Finally, the equality $V_1 = V_2$ is a direct consequence of the following general fact: for any two proper ideals $V_1 \subseteq V_2 \subseteq \mathcal{P}_N$, if there exists a linearly V_2-independent set $B \subseteq \mathcal{P}_N$ which is simultaneously a V_1-basis of $\mathcal{P}_N$ (in our case $B = \bigcup_{k=0}^{\varkappa} \Sigma_k^{V_2}$), then $V_1 = V_2$. □

In the following lemma we indicate a column relation between Σ_{k-1}^V and Σ_k^V.

Lemma 5. *If V is a proper ideal in $\mathcal{P}_N$, then for every $k \in \overline{1, \varkappa_V}$, there exists a column polynomial R_k and an injective scalar matrix M_k such that*

$$\begin{bmatrix} X_1 \Sigma_{k-1}^V \\ \vdots \\ X_N \Sigma_{k-1}^V \end{bmatrix} \overset{V}{=} M_k \Sigma_k^V + R_k \quad \textit{and} \quad \deg R_k < k. \tag{11}$$

The matrix M_k appearing in (11) *is unique.*

Proof. It follows from (9) and Lemma 3 that there exists a column polynomial R_k and a unique scalar matrix M_k which satisfy (11). Since, by (10), all the entries of the column Σ_k^V appear among entries of columns $X_1\Sigma_{k-1}^V, \ldots, X_N\Sigma_{k-1}^V$, we conclude that the matrix M_k contains rows [1, 0, . . . , 0], [0, 1, 0, . . . , 0], ..., [0, . . . , 0, 1]. As a consequence, M_k is injective. □

The next result relates some V-bases of $\mathcal{P}_N$ to the canonical V-basis Λ_N^V.

Proposition 6. *Let V be a proper ideal in $\mathcal{P}_N$ and $\{Q_k\}_{k=0}^{\varkappa_V}$ be a column representation of a non-empty subset B of $\mathcal{P}_N$ such that*

$$Q_k \subseteq \mathcal{P}_N^{\langle k \rangle}, \quad k \in \overline{0, \varkappa_V}. \tag{12}$$

Then the following conditions are equivalent:

(i) *B is linearly V-independent and for every* $k \in \overline{0, \varkappa_V}$, *the length of* Q_k *is equal to* $d_V(k)$,

(ii) *for every* $k \in \overline{0, \varkappa_V}$, $B \cap \mathcal{P}_N^{\langle k]}$ *is a V-basis of* $\mathcal{P}_N^{\langle k]}$ *and*

$$Q_k \subseteq \{p \in \mathcal{P}_N : \deg p = k\}, \tag{13}$$

(iii) *for every* $k \in \overline{0, \varkappa_V}$, *there exists a non-singular scalar matrix* G_k *and a column polynomial* R_k *such that* $Q_k \overset{V}{=} G_k \Sigma_k^V + R_k$ *and* $\deg R_k < k$.

Moreover, if (i) *holds, then* $\{Q_k\}_{k=0}^{\varkappa_V}$ *is a V-basis of* $\mathcal{P}_N$.

Proof. (i)⇒(ii) Fix $k \in \overline{0, \varkappa_V}$. By (12), we have

$$\text{card } B \cap \mathcal{P}_N^{\langle k]} \geqslant \ell(Q_0) + \cdots + \ell(Q_k) = d_V(0) + \cdots + d_V(k) = \dim \Pi_V(\mathcal{P}_N^{\langle k]}).$$

Since the set $B \cap \mathcal{P}_N^{\langle k]}$ is linearly V-independent, we conclude that $B \cap \mathcal{P}_N^{\langle k]}$ is a V-basis of $\mathcal{P}_N^{\langle k]}$ and

$$\text{card } B \cap \mathcal{P}_N^{\langle k]} = \ell(Q_0) + \cdots + \ell(Q_k) = \dim \Pi_V(\mathcal{P}_N^{\langle k]}). \tag{14}$$

We now prove (13). By (12), (13) holds for $k = 0$. Suppose that, contrary to our claim, there are $k \in \mathbb{N}$ such that $k + 1 \leqslant \varkappa_V$, and $p \in Q_{k+1}$ such that $\deg p \neq k + 1$. Then, by (12), $\deg p \leqslant k$. Hence $p \in \left(B \cap \mathcal{P}_N^{\langle k]}\right) \setminus \bigcup_{i=0}^{k} Q_i$ and, in consequence, $\text{card } B \cap \mathcal{P}_N^{\langle k]} > \ell(Q_0) + \cdots + \ell(Q_k)$ which contradicts (14).

(ii)⇒(iii) It follows from (13) that $B \cap \mathcal{P}_N^{\langle k]} = \bigcup_{i=0}^{k} Q_i$ and consequently that $\{Q_i\}_{i=0}^{k}$ is a V-basis of $\mathcal{P}_N^{\langle k]}$ for all $k \in \overline{0, \varkappa_V}$. This enables us to show that $\text{card } Q_k = d_V(k)$ for all $k \in \overline{0, \varkappa_V}$. Fix $k \in \overline{0, \varkappa_V}$. Applying (9) and Lemma 3 to $Y_i = \Sigma_i^V$, $i = 0, \ldots, k$, and $F = \mathcal{P}_N^{\langle k]}$, we find a square scalar matrix G_k and a column polynomial R_k such that $Q_k \overset{V}{=} G_k \Sigma_k^V + R_k$ and $\deg R_k < k$. Likewise, applying Lemma 3 to $Y_i = Q_i$, $i = 0, \ldots, k$, and $F = \mathcal{P}_N^{\langle k]}$, we find a square scalar matrix G'_k and a column polynomial R'_k such that $\Sigma_k^V \overset{V}{=} G'_k Q_k + R'_k$ and $\deg R'_k < k$. Since $Q_k \overset{V}{=} G_k \Sigma_k^V + R_k$, we get $\Sigma_k^V \overset{V}{=} (G'_k G_k) \Sigma_k^V + (G'_k R_k + R'_k)$ and $\deg(G'_k R_k + R'_k) < k$. Hence, once more by (9) and Lemma 3 applied to $Y_i = \Sigma_i^V$, $i = 0, \ldots, s$, and $F = \mathcal{P}_N^{\langle s]}$ with $s = k - 1, k$, we see that $G'_k G_k$ is the identity matrix.

(iii)⇒(i) Notice that for every $k \in \overline{0, \varkappa_V}$, $\ell(Q_k) = d_V(k)$ (because $Q_k \overset{V}{=} G_k \Sigma_k^V + R_k$, $\ell(\Sigma_k^V) = d_V(k)$ and G_k is a square matrix) and

$$\text{if } A_i \text{ are scalar rows such that } \sum_{i=0}^{k} A_i Q_i \overset{V}{=} 0, \text{ then } A_i = 0 \text{ for } i = 0, \ldots, k. \tag{15}$$

The proof of (15) is by induction on k. The case $k = 0$ is easily seen to be true. Suppose (15) is valid for a fixed integer $0 \leqslant k < \varkappa_V$, and $\sum_{i=0}^{k+1} A_i Q_i \overset{V}{=} 0$. Since $Q_{k+1} \overset{V}{=} G_{k+1} \Sigma_{k+1}^V + R_{k+1}$, we get $A_{k+1} G_{k+1} \Sigma_{k+1}^V + R'_{k+1} \overset{V}{=} 0$ for some $R'_{k+1} \in \mathcal{P}_N^{\langle k]}$ (use (12)). Hence, by (9) and Lemma 3 applied to $Y_i = \Sigma_i^V$, $i = 0, \ldots, s$, and $F = \mathcal{P}_N^{\langle s]}$ with $s = k, k + 1$, we conclude that $A_{k+1}G_{k+1} = 0$, which, together with the non-singularity of G_{k+1}, gives us $A_{k+1} = 0$. This proves (15), which in turn implies the linear V-independence of B (cf. Lemma 3).

To prove the last assertion of Proposition 6, we have to show that $\mathcal{P}_N$ is a linear V-span of $\{Q_k\}_{k=0}^{\varkappa_V}$ provided (i) holds. By Propositions 1 and 2, $\mathcal{P}_N$ is a linear V-span of $\{\Sigma_k^V\}_{k=0}^{\varkappa_V}$. Hence, it is sufficient to prove that for every $k \in \overline{0, \varkappa_V}$, $\Pi_V(\bigcup_{i=0}^k \Sigma_i^V) \subseteq \operatorname{lin} \Pi_V(\bigcup_{i=0}^k Q_i)$. We proceed by induction on k. The case $k = 0$ is obvious. Suppose the induction hypothesis is true for a fixed integer $0 \leqslant k < \varkappa_V$. It follows from (iii) that $\Sigma_{k+1}^V \overset{V}{=} G_{k+1}^{-1} Q_{k+1} - G_{k+1}^{-1} R_{k+1}$. Since $\deg G_{k+1}^{-1} R_{k+1} \leqslant k$, we infer from (9) that there are scalar matrices $A_0, \ldots, A_k$ such that

$$\Sigma_{k+1}^V \overset{V}{=} G_{k+1}^{-1} Q_{k+1} + \sum_{i=0}^{k} A_i \Sigma_i^V,$$

which by the induction hypothesis completes the proof. □

Remark 7. It is worthwhile to point out the role played by (13) in Proposition 6. Suppose that $\varkappa_V \geqslant 2$ and $d_V(j) \geqslant 2$ for some $0 \leqslant j < \varkappa_V$ (e.g. $V = \{0\} \subseteq \mathcal{P}_2$). We know that $\{\Sigma_k^V\}_{k=0}^{\varkappa_V}$ satisfies condition (ii) of Proposition 6 (with $B = \Lambda_N^V$). Set $Q_k = \Sigma_k^V$ for all $k \neq j, j+1$. Remove an entry p from Σ_j^V and denote the so-obtained column by Q_j. Next attach p to the column Σ_{j+1}^V (as an extra entry) and denote the so-obtained column by Q_{j+1}. Then for every $k \in \overline{0, \varkappa_V}$, $\mathcal{P}_N^{\langle k]} \cap \bigcup_{i=0}^{\varkappa_V} Q_k$ is a V-basis of $\mathcal{P}_N^{\langle k]}$. However $\{Q_k\}_{k=0}^{\varkappa_V}$ does not satisfy (13). In other words, condition (13) distinguishes column representations of a given V-basis of $\mathcal{P}_N$ in which the index of each column coincides with the degree of every entry of this column.

We conclude this section with a (relatively) simple method of producing V-bases of $\mathcal{P}_N$ satisfying the assumptions of Proposition 6.

Proposition 8. *Let V be a proper ideal in $\mathcal{P}_N$ and let C be a subset of $\mathcal{P}_N$ such that $\operatorname{lin} C \cap \mathcal{P}_N^{\langle k]} = \mathcal{P}_N^{\langle k]}$ for every integer $k \geqslant 0$. If the set $B \overset{\text{df}}{=} C \setminus V$ is linearly V-independent, then B has a column representation $\{Q_k\}_{k=0}^{\varkappa_V}$ which satisfies conditions* (12) *and* (i) *of Proposition* 6.

Proof. Since the set $B \cap \mathcal{P}_N^{\langle k]}$ is linearly V-independent and

$$\Pi_V(\mathcal{P}_N^{\langle k]}) = \operatorname{lin} \Pi_V(C \cap \mathcal{P}_N^{\langle k]}) = \operatorname{lin} \Pi_V(B \cap \mathcal{P}_N^{\langle k]}),$$

we conclude that for every integer $k \geqslant 0$, the set $B \cap \mathcal{P}_N^{\langle k]}$ is a V-basis of $\mathcal{P}_N^{\langle k]}$. This in turn implies that $\operatorname{card}\{p \in B : \deg p = 0\} = \operatorname{card} B \cap \mathcal{P}_N^{\langle 0]} = d_V(0)$ and

$$\operatorname{card}\{p \in B : \deg p = k\} = \operatorname{card} B \cap \mathcal{P}_N^{\langle k]} - \operatorname{card} B \cap \mathcal{P}_N^{\langle k-1]} = d_V(k), \quad k \geqslant 1.$$

Arranging members of the set $\{p \in B : \deg p = k\}$ in a column Q_k of length $d_V(k)$ (in an arbitrary way), we get the required column representation $\{Q_k\}_{k=0}^{\varkappa_V}$ of B.

4. Rigid V-bases

Let V be a proper ideal in $\mathcal{P}_N$. A sequence $\{P_k\}_{k=0}^{\varkappa_V}$ of column polynomials is said to be a *rigid* V-*basis* of $\mathcal{P}_N$, if $\{P_k\}_{k=0}^{\varkappa_V}$ is a column representation of a V-basis of $\mathcal{P}_N$ such that for every $k \in \overline{0, \varkappa_V}$, $P_k \subseteq \mathcal{P}_N^{\langle k \rangle}$ and $\ell(P_k) = d_V(k)$. If $V = \{0\}$, we call it simply a *rigid basis* of $\mathcal{P}_N$; notice that $\varkappa_{\{0\}} = \infty$ and

$$d_{\{0\}}(k) = \mathrm{card}\{\alpha \in \mathbb{N}^N : |\alpha| = k\} = \binom{k+N-1}{k}, \quad k \geqslant 0.$$

If $\{P_k\}_{k=0}^{\varkappa_V}$ is a rigid V-basis of $\mathcal{P}_N$, then, by Propositions 6, for every $k \in \overline{0, \varkappa_V}$, $\bigcup_{i=0}^{k} P_i$ is a V-basis of $\mathcal{P}_N^{\langle k \rangle}$ and the degree of each member of P_k is equal to k. Moreover, if $\{P_k\}_{k=0}^{\varkappa_V}$ and $\{Q_k\}_{k=0}^{\varkappa_V}$ are rigid V-basis of $\mathcal{P}_N$ and $\bigcup_{i=0}^{\varkappa_V} P_i = \bigcup_{i=0}^{\varkappa_V} Q_i$, then for every $k \in \overline{0, \varkappa_V}$, the columns P_k and Q_k are identical up to an arrangement of entries.

It is possible to construct a V-basis B of $\mathcal{P}_N$, none of whose column representations is a rigid V-basis of $\mathcal{P}_N$. What is more, B may be chosen so that $\{\deg p : p \in B\}$ is an arbitrary cofinal subset of $\overline{0, \varkappa_V}$ (“cofinal” means that for every $k \in \overline{0, \varkappa_V}$ there exists $p \in B$ such that $\deg p \geqslant k$). This can be done with the help of the following auxiliary fact applied to $C = \bigcup_{k=0}^{\varkappa_V} \Sigma_k^V$ (appropriately partitioned): if a basis C of a complex vector space F is equal to the union $\bigcup_{\omega \in \Omega} C_\omega$ of pairwise disjoint non-empty subsets of C, and $v_\omega \in C_\omega$ for all $\omega \in \Omega$, then the sets $C_\omega + v_\omega$, $\omega \in \Omega$, are pairwise disjoint and $\bigcup_{\omega \in \Omega}(C_\omega + v_\omega)$ is a basis of F. In particular, if $\varkappa_V < \infty$, then it is possible to find a V-basis of $\mathcal{P}_N$ composed of polynomials of degree $\varkappa_V$.

Owing to (9), the sequence $\{\Sigma_k^V\}_{k=0}^{\varkappa_V}$ defined in Section 2 is a rigid V-basis of $\mathcal{P}_N$. Hence, if P is a column polynomial, then, by Lemma 3, there exists a unique finite sequence $\{D_k\}_{k=0}^{\varkappa_V}$ of scalar matrices such that $P \overset{V}{=} \sum_{k=0}^{\varkappa_V} D_k \Sigma_k^V$; the scalar matrix D_k is called the *kth coefficient* of P (relative to V) and is denoted by $P_{[k]}$. For simplicity of notation, we do not indicate the dependence of $P_{[k]}$ on the ideal V; this will cause no confusion.

The following fact derived from Proposition 6 is a useful criterion for rigidity of V-bases.

Proposition 9. *Let V be a proper ideal in $\mathcal{P}_N$. A sequence $\{Q_k\}_{k=0}^{\varkappa_V}$ of column polynomials is a rigid V-basis of $\mathcal{P}_N$ if and only if the following two conditions hold for every $k \in \overline{0, \varkappa_V}$:*

(a) $\deg q \leqslant k$ *for every entry q of Q_k;*
(b) *there exists a non-singular scalar matrix G_k and a column polynomial R_k such that $Q_k \overset{V}{=} G_k \Sigma_k^V + R_k$ and* $\deg R_k < k$.

Proof. The “only if” part of the conclusion is an immediate consequence of Proposition 6. A careful inspection of the proof of the implication (iii)⇒(i) in Proposition 6 shows that conditions (a) and (b) imply (15). This, when combined with Lemma 3, implies that $\{Q_k\}_{k=0}^{\varkappa_V}$ is a column representation of a linearly V-independent set. Applying Proposition 6 completes the proof. □

As is shown below every rigid V-basis of $\mathcal{P}_N$ may be enlarged by members of V so as to get a rigid basis of $\mathcal{P}_N$.

Proposition 10. *Let V be a proper ideal in $\mathcal{P}_N$. Then there exists a sequence $\{T_k\}_{k=0}^{\infty}$ of (possibly empty) subsets of V such that for every rigid V-basis $\{Q_k\}_{k=0}^{\varkappa_V}$ of $\mathcal{P}_N$, the sequence $\{P_k\}_{k=0}^{\infty}$ of column polynomials defined by*[3]

$$P_k = \begin{cases} Q_k \cup T_k & \text{for } k \in \overline{0, \varkappa_V}, \\ T_k & \text{for } k > \varkappa_V, \end{cases} \tag{16}$$

is a rigid basis of $\mathcal{P}_N$.

Remark 11. Notice that, if $\{Q_k\}_{k=0}^{\varkappa_V}$ is a rigid V-basis of $\mathcal{P}_N$ and $\{T_k\}_{k=0}^{\infty}$ is a sequence of subsets of V such that $\{P_k\}_{k=0}^{\infty}$ defined by (16) is a rigid basis of $\mathcal{P}_N$, then for every $k \geqslant 0$, $T_k \subseteq \{p \in V : \deg p = k\}$ and $\bigcup_{i=0}^{k} T_i$ is a basis of $V \cap \mathcal{P}_N^{\langle k \rangle}$.

Proof of Proposition 10. Since $\{V \cap \mathcal{P}_N^{\langle k \rangle}\}_{k=0}^{\infty}$ is an increasing sequence of linear spaces, there exists a sequence $\{T_k\}_{k=0}^{\infty}$ of subsets of V such that for every $k \geqslant 0$, $\bigcup_{i=0}^{k} T_i$ is a basis of $V \cap \mathcal{P}_N^{\langle k \rangle}$ and $T_{k+1} \cap \bigcup_{i=0}^{k} T_i = \varnothing$ (use recursion beginning with $T_0 = \varnothing$ as $V \cap \mathcal{P}_N^{\langle 0 \rangle} = \{0\}$). Clearly, we have $T_k \subseteq \{p \in V : \deg p = k\}$ for every $k \geqslant 0$. Notice that $T_k \neq \varnothing$ for every $k > \varkappa_V$. Indeed, otherwise $T_k = \varnothing$ for some $k > \varkappa_V$. Let $\alpha \in \mathbb{N}^N$ be such that $|\alpha| = k$. Since $\{\Sigma_i^V\}_{i=0}^{\varkappa_V}$ is a rigid V-basis of $\mathcal{P}_N$, there exists a sequence $\{D_i\}_{i=0}^{\varkappa_V}$ of scalar rows such that $X^\alpha - \sum_{i=0}^{\varkappa_V} D_i \Sigma_i^V \in V \cap \mathcal{P}_N^{\langle k \rangle} = \operatorname{lin} \bigcup_{i=0}^{k-1} T_i$, which is impossible.

We claim that $\{T_k\}_{k=0}^{\infty}$ is the required sequence. First we show that $\{P_k\}_{k=0}^{\infty}$ is linearly independent. Let $\{q_j\}_{j \in J} \subseteq \bigcup_{k=0}^{\varkappa_V} Q_k$ and $\{t_j\}_{j \in K} \subseteq \bigcup_{k=0}^{\infty} T_k$ be finite systems of pairwise distinct polynomials such that

$$\sum_{j \in J} a_j q_j + \sum_{j \in K} b_j t_j = 0, \tag{17}$$

where $\{a_j\}_{j \in J}$ and $\{b_j\}_{j \in K}$ are finite sequences of complex numbers. Then clearly $\sum_{j \in J} a_j q_j \overset{V}{=} 0$, which implies $a_j = 0$ for all $j \in J$. By (17), we get $\sum_{j \in K} b_j t_j = 0$, and consequently $b_j = 0$ for all $j \in K$. Hence $\{P_k\}_{k=0}^{\infty}$ is linearly independent.

If $p \in \mathcal{P}_N^{\langle k \rangle}$ $(k \geqslant 0)$, then by rigidity of the V-basis $\{Q_j\}_{j=0}^{\varkappa_V}$ there exist finite systems $\{q_j\}_{j \in J} \subseteq \bigcup_{i=0}^{s} Q_i$ and $\{a_j\}_{j \in J} \subseteq \mathbb{C}$ such that $p - \sum_{j \in J} a_j q_j \in V \cap \mathcal{P}_N^{\langle k \rangle}$, where $s \overset{\text{df}}{=} \min\{k, \varkappa_V\}$. Since $V \cap \mathcal{P}_N^{\langle k \rangle} = \operatorname{lin} \bigcup_{j=0}^{k} T_j$, there exist finite systems $\{t_j\}_{j \in K} \subseteq \bigcup_{i=0}^{k} T_i$ and $\{b_j\}_{j \in K} \subseteq \mathbb{C}$ such that $p = \sum_{j \in J} a_j q_j + \sum_{j \in K} b_j t_j$. This shows that $\mathcal{P}_N^{\langle k \rangle}$ is the linear span of $\{P_j\}_{j=0}^{k}$. Hence $\{P_j\}_{j=0}^{k}$ is a basis of $\mathcal{P}_N^{\langle k \rangle}$ for every $k \in \mathbb{N}$, and consequently $\{P_k\}_{k=0}^{\infty}$ is a rigid basis of $\mathcal{P}_N$. □

[3] Entries of each column P_k are distinct and Q_i is a subcolumn of P_i for every $i \in \overline{0, \varkappa_V}$.

5. Quasi-orthogonality: the real case

In this section we focus attention on the relationship between the quasi-orthogonality with respect to a linear functional $L : \mathcal{P}_N \to \mathbb{C}$ and the three term relations modulo a $*$-ideal in $\mathcal{P}_N$ (recall that an ideal V in $\mathcal{P}_N$ is said to be a *$*$-ideal*, if $p^* \in V$ for all $p \in V$). A sequence $\{Q_k\}_{k=0}^n$ $(0 \leqslant n \leqslant \infty)$ of column polynomials is said to be *quasi-orthogonal* with respect to L if $L(Q_i Q_j^*) = 0$ for all $i \neq j$.

Given a non-empty subset J of $\mathbb{N}$ and a system $\{Q_k\}_{k \in J}$ of column polynomials, we say that $\{Q_k\}_{k \in J}$ is *selected* from a sequence $\{P_k\}_{k=0}^\infty$ of column polynomials (briefly: $\{Q_k\}_{k \in J} \preccurlyeq \{P_k\}_{k=0}^\infty$) if for every $k \in J$, the column Q_k is made up of P_k by removing some entries of P_k and leaving the remainder in the order inherited from P_k. Given a linear functional $L : \mathcal{P}_N \to \mathbb{C}$, we define the set

$$\mathcal{V}_L = \bigcap_{q \in \mathcal{P}_N} \{p \in \mathcal{P}_N : L(pq) = 0\}.$$

It is clear that $\mathcal{V}_L$ is an ideal in $\mathcal{P}_N$ such that $\mathcal{V}_L \subseteq \ker L$. The latter inclusion and the definition of $\mathcal{V}_L$ imply that $\mathcal{V}_L$ is the greatest ideal contained in $\ker L$, and that $\mathcal{V}_L$ is a proper ideal if and only if L is non-zero. If L is a Hermitian linear functional, then $\mathcal{V}_L$ is a $*$-ideal. Example 12 shows that $\mathcal{V}_L$ may be $*$-ideal, though L is not a non-zero scalar multiple of a Hermitian linear functional.

Example 12. Let μ be a positive Borel measure on $\mathbb{R}^N$. Assume that the closed support $\operatorname{supp} \mu$ of μ is compact and it has a non-empty interior. Take two linearly independent polynomials $\varphi_1, \varphi_2 \in \mathcal{R}_N$. Set $\varphi = \varphi_1 + \mathrm{i}\varphi_2$ and

$$L(p) = \int_{\mathbb{R}^N} \varphi p \, \mathrm{d}\mu, \quad p \in \mathcal{P}_N.$$

Then $\mathcal{V}_L = \{0\}$ (which, of course, is a $*$-ideal). Indeed, if $p \in \mathcal{V}_L$, then $p = p_1 + \mathrm{i}p_2$, where $p_1, p_2 \in \mathcal{R}_N$, and

$$0 = L(pq) = \int_{\mathbb{R}^N} (\varphi_1 p_1 - \varphi_2 p_2) q \, \mathrm{d}\mu + \mathrm{i} \int_{\mathbb{R}^N} (\varphi_2 p_1 + \varphi_1 p_2) q \, \mathrm{d}\mu$$

for every $q \in \mathcal{R}_N$. Using the non-emptiness of the interior of $\operatorname{supp} \mu$ and the uniqueness theorem for polynomials, we get $\varphi_1 p_1 = \varphi_2 p_2$ and $\varphi_2 p_1 = -\varphi_1 p_2$. This in turn implies that $\varphi_1 \varphi_2 (p_1^2 + p_2^2) = 0$. Since the polynomials φ_1 and φ_2 are linearly independent, the product $\varphi_1 \varphi_2$ is non-zero, and consequently $p_1^2 + p_2^2 = 0$. Hence $p = 0$, which means that $\mathcal{V}_L = \{0\}$.

Suppose that, contrary to our claim, there exists a non-zero complex number $z = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$ such that zL is Hermitian. Then for any $p \in \mathcal{R}_N$, we have $\overline{zL(p)} = zL(p)$, which gives us $\int_{\mathbb{R}^N} (x\varphi_2 + y\varphi_1) p \, \mathrm{d}\mu = 0$. Arguing as in the previous paragraph, we show that $x\varphi_2 + y\varphi_1 = 0$, which contradicts the linear independence of φ_1 and φ_2.

Proposition 13. *Let $\{P_k\}_{k=0}^\infty$ be a rigid basis of $\mathcal{P}_N$ and $L : \mathcal{P}_N \to \mathbb{C}$ be a linear functional such that $L(P_i P_j^*) = 0$ for all $i \neq j$. If $V \overset{\mathrm{df}}{=} \mathcal{V}_L$ is a proper $*$-ideal,* *then*

(i) *the rank of the matrix* $L(P_k P_k^*)$ *is equal to* $d_V(k)$ *for every integer* $k \geqslant 0$;

(ii) *if* $\{Q_k\}_{k\in J} \preccurlyeq \{P_k\}_{k=0}^{\infty}$, *then* $\bigcup_{k\in J} Q_k$ *is a V-basis of* $\mathcal{P}_N$ *if and only if* $J = \overline{0, \varkappa_V}$ *and* $L(Q_k Q_k^*)$ *is a non-singular* $d_V(k) \times d_V(k)$*-matrix for every* $k \in J$; *moreover, there always exists such a V-basis of* $\mathcal{P}_N$;

(iii) *every V-basis* $\{Q_k\}_{k=0}^{\varkappa_V}$ *of* $\mathcal{P}_N$ *selected from* $\{P_k\}_{k=0}^{\infty}$ *is rigid*;

(iv) *if* $\{Q_k\}_{k=0}^{\varkappa_V}$ *is a V-basis of* $\mathcal{P}_N$ *selected from* $\{P_k\}_{k=0}^{\infty}$, *then for every* $k \in \overline{0, \varkappa_V}$, *there exists a* (*unique*) *system* $A_{k,1}, \ldots, A_{k,N}, B_{k,1}, \ldots, B_{k,N}, C_{k,1}, \ldots, C_{k,N}$ *of scalar matrices such that*

  (iv-a) $X_j Q_k \overset{V}{=} A_{k,j} Q_{k+1} + B_{k,j} Q_k + C_{k,j} Q_{k-1}$ *for all* $j = 1, \ldots, N$, *where* $C_{0,j} \overset{\text{df}}{=} 1$ *and* $Q_{-1} \overset{\text{df}}{=} 0$; *if* $\varkappa_V < \infty$, *then* $A_{\varkappa_V, j} \overset{\text{df}}{=} [1, \ldots, 1]^*$ *with the number of entries equal to the length of* $Q_{\varkappa_V}$ *and* $Q_{\varkappa_V+1} \overset{\text{df}}{=} 0$,

  (iv-b) *the matrices* $[A_{k,1}^*, \ldots, A_{k,N}^*]^*$ *and* $[C_{k,1}, \ldots, C_{k,N}]$ *are of maximal rank.*

Proof. We first show that for every column polynomial W,

$$W \overset{V}{=} 0 \quad \text{if and only if} \quad L(W P_j^*) = 0 \quad \text{for every integer } j \geqslant 0. \tag{18}$$

The "only if" part is obvious. Suppose $L(W P_j^*) = 0$ for all $j \geqslant 0$. Since $\{P_j\}_{j=0}^{\infty}$ is a basis of $\mathcal{P}_N$, Lemma 3 yields $L(W Q^*) = 0$ for every column polynomial Q (of arbitrary length). This implies $W \overset{V}{=} 0$ because $V = \mathcal{V}_L$.

Next we prove that

$$\text{for every integer } k \geqslant 0,\ L(P_k P_k^*) = 0 \quad \text{if and only if} \quad k > \varkappa_V. \tag{19}$$

If $L(P_k P_k^*) = 0$, then by the quasi-orthogonality assumption $L(P_k P_j^*) = 0$ for all $j \geqslant 0$. Hence, by (18), we have $\Pi_V(P_k) = 0$, which in turn implies

$$\begin{aligned} d_V(k) &= \dim\ \Pi_V(\mathcal{P}_N^{\langle k]}) - \dim\ \Pi_V(\mathcal{P}_N^{\langle k-1]}) \\ &= \dim\ \mathrm{lin} \bigcup_{i=0}^{k} \Pi_V(P_i) - \dim\ \mathrm{lin} \bigcup_{i=0}^{k-1} \Pi_V(P_i) = 0 \quad \text{provided } k \geqslant 1. \end{aligned}$$

This means that $k > \varkappa_V$ (the case $k = 0$ never happens, because $1 = d_V(0) = \dim\ \Pi_V(\mathcal{P}_N^{\langle 0]}) = \dim\ \mathrm{lin}\, \Pi_V(P_0)$). Conversely, if $k > \varkappa_V$, then by Lemma 3 and Propositions 1 and 2 there are scalar matrices[4] $E_0, \ldots, E_{\varkappa_V}$ and $F_0, \ldots, F_{\varkappa_V}$ such that $P_k \overset{V}{=} \sum_{j=0}^{\varkappa_V} E_j \Sigma_j^V$ and $\sum_{j=0}^{\varkappa_V} E_j \Sigma_j^V = \sum_{j=0}^{\varkappa_V} F_j P_j$ (because $\{P_i\}_{i=0}^{\infty}$ is a rigid basis of $\mathcal{P}_N$), which yields $L(P_k P_k^*) = \sum_{j=0}^{\varkappa_V} F_j L(P_j P_k^*) = 0$.

(iii) Let $\{Q_k\}_{k=0}^{\varkappa_V}$ be a V-basis of $\mathcal{P}_N$ selected from $\{P_k\}_{k=0}^{\infty}$. By Lemma 3, for every $k \in \overline{0, \varkappa_V}$, there exists a unique finite sequence $\{D_{k,j}\}_{j=0}^{\varkappa_V}$ of scalar matrices such that

$$P_k \overset{V}{=} \sum_{j=0}^{\varkappa_V} D_{k,j} Q_j. \tag{20}$$

[4] For simplicity, we suppress the explicit dependence of E_i and F_i on k in the notation.

We first show that

$$\text{the matrix } L(Q_k Q_k^*) \text{ is non-singular for every } k \in \overline{0, \varkappa_V}. \tag{21}$$

Indeed, otherwise $L(Q_k Q_k^*)$ is singular for some such k. Then there exists a scalar row $a_k \neq 0$ of appropriate length such that $a_k L(Q_k Q_k^*) = 0$. By the quasi-orthogonality assumption, we have $L(a_k Q_k P_j^*) = a_k L(Q_k P_j^*) = 0$ for all $j \in \mathbb{N} \setminus \{k\}$. Since V is a $*$-ideal, we infer from (20) that $L(a_k Q_k P_k^*) = a_k L(Q_k Q_k^*) D_{k,k}^* = 0$. By (18), $a_k Q_k \overset{V}{=} 0$, which contradicts the linear V-independence of $\{Q_i\}_{i=0}^{\varkappa_V}$. This proves (21). Next we show that

$$P_k \overset{V}{=} D_{k,k} Q_k, \quad k \in \overline{0, \varkappa_V}. \tag{22}$$

Indeed, it follows from (20) and the quasi-orthogonality assumption that

$$0 = L(P_k Q_i^*) = D_{k,i} L(Q_i Q_i^*), \quad i \in \overline{0, \varkappa_V} \setminus \{k\}.$$

Since $L(Q_i Q_i^*)$ is non-singular, we get $D_{k,i} = 0$ for all $i \in \overline{0, \varkappa_V} \setminus \{k\}$. This and (20) yield (22). Set $B = \bigcup_{i=0}^{\varkappa_V} Q_i$. Since $\{P_i\}_{i=0}^{\infty}$ is a rigid basis of $\mathcal{P}_N$, we infer from (22) that $B \cap \mathcal{P}_N^{\langle k \rangle} = \bigcup_{i=0}^{k} Q_i$ is a V-basis of $\mathcal{P}_N^{\langle k \rangle}$ for every $k \in \overline{0, \varkappa_V}$. By part (ii) of Proposition 6, $\{Q_k\}_{k=0}^{\varkappa_V}$ is a rigid V-basis of $\mathcal{P}_N$.

(i) & (ii) Since $\mathcal{P}_N$ is the linear span of $\{P_k\}_{k=0}^{\infty}$, there exists a system $\{Q_k\}_{k \in J} \preccurlyeq \{P_k\}_{k=0}^{\infty}$ such that $\bigcup_{k \in J} Q_k$ is a V-basis of $\mathcal{P}_N$. Let now $\{Q_k\}_{k \in J}$ be any such system. We first show that $\overline{0, \varkappa_V} \subseteq J$. Suppose that, contrary to our claim, there exists $k \in \overline{0, \varkappa_V} \setminus J$. We know that there exists a finite system $\{D_j\}_{j \in J}$ of scalar matrices such that $P_k \overset{V}{=} \sum_{j \in J} D_j Q_j$. This and $k \notin J$ give us $L(P_k P_k^*) = \sum_{j \in J} D_j L(Q_j P_k^*) = 0$, which contradicts (19). Suppose now that $k \in J$ and $k > \varkappa_V$. Then, by (19), $L(Q_k P_j^*) = 0$ for all integers $j \geqslant 0$. Applying (18), we obtain $Q_k \overset{V}{=} 0$, which contradicts the linear V-independence of $\{Q_i\}_{i \in J}$. This means that $J = \overline{0, \varkappa_V}$. According to (iii), $\{Q_k\}_{k \in J}$ is a rigid V-basis of $\mathcal{P}_N$. Therefore, by (21), for every $k \in J$, $L(Q_k Q_k^*)$ is a non-singular $d_V(k) \times d_V(k)$-submatrix of $L(P_k P_k^*)$ (because $\{Q_i\}_{i \in J} \preccurlyeq \{P_i\}_{i=0}^{\infty}$). This implies that $d_V(k) \leqslant \operatorname{rank} L(P_k P_k^*)$ for all $k \in J$. It follows from (22) that for every $k \in J$, $L(P_k P_k^*) = D_{k,k} L(Q_k P_k^*)$, and so $\operatorname{rank} L(P_k P_k^*) \leqslant \operatorname{rank} D_{k,k} \leqslant d_V(k)$ (because $D_{k,k}$ has $d_V(k)$ columns). This and (19) give us $\operatorname{rank} L(P_k P_k^*) = d_V(k)$ for all $k \in \mathbb{N}$. Summarizing, we have proved (i) and the "only if" part of (ii).

Suppose now that $\{Q_k\}_{k=0}^{\varkappa_V} \preccurlyeq \{P_k\}_{k=0}^{\infty}$ is such that for every $k \in \overline{0, \varkappa_V}$, $L(Q_k Q_k^*)$ is a non-singular $d_V(k) \times d_V(k)$-matrix. By our assumption, we have

$$L(Q_i Q_j^*) = 0 \quad \text{for all } i, j \in \overline{0, \varkappa_V} \text{ such that } i \neq j. \tag{23}$$

Fix $n \in \overline{0, \varkappa_V}$. Suppose that $D_1, \ldots, D_n$ are scalar rows such that $\sum_{j=0}^{n} D_j Q_j \overset{V}{=} 0$. Multiplying both sides of the equality by Q_k^* and applying L to the result we get $D_k L(Q_k Q_k^*) = 0$, and so $D_k = 0$ for $k = 0, \ldots, n$. Hence, the sequence $\{Q_k\}_{k=0}^{\varkappa_V}$ is linearly V-independent. By part (i) of Proposition 6, $\{Q_k\}_{k=0}^{\varkappa_V}$ is a rigid V-basis of $\mathcal{P}_N$.

(iv) Let $\{Q_k\}_{k=0}^{\varkappa_V}$ be a V-basis of $\mathcal{P}_N$ selected from $\{P_k\}_{k=0}^{\infty}$. Fix $j \in \{1, \ldots, N\}$. Consider first the case $\varkappa_V = \infty$. By (iii) and Lemma 3, for every $k \in \overline{0, \varkappa_V}$, there exists a unique system of scalar matrices $D_0^{(k)}, \ldots, D_{k+1}^{(k)}$ (for simplicity of notation, we do not make its dependence on j explicit) such that

$$X_j Q_k \overset{V}{=} \sum_{i=0}^{k+1} D_i^{(k)} Q_i. \tag{24}$$

Since V is a $*$-ideal, we infer from (23) and (24) (with $k = l$) that

$$L(X_j Q_k Q_l^*) = L(Q_k (X_j Q_l)^*) = \sum_{i=0}^{l+1} L(Q_k Q_i^*)(D_i^{(l)})^* = 0$$

for all $k, l \in \overline{0, \varkappa_V}$ such that $k \geqslant l + 2$. This, when combined with (23) and (24), leads to

$$0 = L(X_j Q_k Q_l^*) = \sum_{i=0}^{k+1} D_i^{(k)} L(Q_i Q_l^*) = D_l^{(k)} L(Q_l Q_l^*)$$

for all $k, l \in \overline{0, \varkappa_V}$ such that $k \geqslant l + 2$. Since, by (ii), the matrix $L(Q_l Q_l^*)$ is non-singular, we get $D_l^{(k)} = 0$ for all $k, l \in \overline{0, \varkappa_V}$ such that $k \geqslant l + 2$. This and (24) imply (iv-a). The case $\varkappa_V < \infty$ can be handled in much the same way (with special care for $k = \varkappa_V$).

Fix an integer $0 \leqslant k < \varkappa_V$ and rewrite condition (iv-a) in the column form

$$\begin{bmatrix} X_1 Q_k \\ \vdots \\ X_N Q_k \end{bmatrix} \overset{V}{=} \begin{bmatrix} A_{k,1} \\ \vdots \\ A_{k,N} \end{bmatrix} Q_{k+1} + \begin{bmatrix} B_{k,1} \\ \vdots \\ B_{k,N} \end{bmatrix} Q_k + \begin{bmatrix} C_{k,1} \\ \vdots \\ C_{k,N} \end{bmatrix} Q_{k-1}. \tag{25}$$

By part (iii) of Proposition 6 and Lemma 5, the $(k + 1)$th coefficient[5] of the column polynomial (relative to V) appearing on the left-hand side of (25) is equal to

$$\left(\begin{bmatrix} G_k & 0 & \cdots & 0 \\ 0 & G_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_k \end{bmatrix} \begin{bmatrix} X_1 \Sigma_k^V \\ X_2 \Sigma_k^V \\ \vdots \\ X_N \Sigma_k^V \end{bmatrix} \right)_{[k+1]} = \begin{bmatrix} G_k & 0 & \cdots & 0 \\ 0 & G_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_k \end{bmatrix} M_{k+1}, \tag{26}$$

while the $(k + 1)$th coefficient of the right-hand side is equal to

$$\begin{bmatrix} A_{k,1} \\ \vdots \\ A_{k,N} \end{bmatrix} G_{k+1}, \tag{27}$$

where G_k and G_{k+1} are as in part (iii) of Proposition 6 and M_{k+1} is as in Lemma 5. Since the right-hand side of (26) coincides with (27), the matrices G_k and G_{k+1} are invertible and the matrix M_{k+1} is injective, we conclude that the matrix $[A_{k,1}^*, \ldots, A_{k,N}^*]^*$ is injective and that its rank is equal to $d_V(k+1)$. If $\varkappa_V < \infty$ and $k = \varkappa_V$, then $[A_{k,1}^*, \ldots, A_{k,N}^*]^* = [1, \ldots, 1]$ with $N d_V(k)$ entries.

[5] See Section 4 for its definition.

We prove that the rank of $[C_{k,1}, \dots, C_{k,N}]$ is maximal. The case $k = 0$ is trivial (because $[C_{0,1}, \dots, C_{0,N}] = [1, \dots, 1]$). Fix an integer $0 \leqslant k < \varkappa_V$. Substituting the three term representations (given by (iv-a)) of columns $X_j Q_{k+1}$ and $X_j Q_k$ into the equality $Q_k (X_j Q_{k+1})^* = (X_j Q_k) Q^*_{k+1}$ (recall that V is a $*$-ideal), then letting the functional L act on both sides of it and simultaneously applying $V \subseteq \ker L$ and (23), we get $L(Q_k Q_k^*) C^*_{k+1,j} = A_{k,j} L(Q_{k+1} Q^*_{k+1})$ for all $j = 1, \dots, N$. This implies

$$\begin{bmatrix} L(Q_k Q_k^*) & 0 & \dots & 0 \\ 0 & L(Q_k Q_k^*) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & L(Q_k Q_k^*) \end{bmatrix} \begin{bmatrix} C^*_{k+1,1} \\ C^*_{k+1,2} \\ \vdots \\ C^*_{k+1,N} \end{bmatrix} = \begin{bmatrix} A_{k,1} \\ A_{k,2} \\ \vdots \\ A_{k,N} \end{bmatrix} L(Q_{k+1} Q^*_{k+1}). \tag{28}$$

By the injectivity of $[A^*_{k,1}, \dots, A^*_{k,N}]^*$ and the non-singularity of $L(Q_{k+1} Q^*_{k+1})$, we infer from (28) that the matrix $[C_{k+1,1}, \dots, C_{k+1,N}]^*$ is injective. Hence the matrix $[C_{k+1,1}, \dots, C_{k+1,N}]$ is of maximal rank. $\square$

Remark 14. Let us clarify the circumstances in which Hermitian linear functionals may appear in Proposition 13.

(a) How to select a V-basis $\{Q_k\}_{k=0}^{\varkappa_V}$ of $\mathcal{P}_N$ from $\{P_k\}_{k=0}^{\infty}$, where $\{P_k\}_{k=0}^{\infty}$ is as in Proposition 13? If all the scalar matrices $L(P_k P_k^*)$ are symmetric (which is the case for a Hermitian L), then the answer is: fix $k \in \overline{0, \varkappa_V}$, choose $d_V(k)$ entries of P_k and arrange them in a column Q_k taking account of the order inherited from P_k; if the rank of the matrix $L(Q_k P_k^*)$ is maximal, then $L(Q_k Q_k^*)$ is a non-singular $d_V(k) \times d_V(k)$-matrix (this is because the matrix $L(P_k P_k^*)$ is symmetric, its submatrix $L(Q_k P_k^*)$ is of maximal rank and rank $L(P_k P_k^*) =$ rank $L(Q_k P_k^*)$). Hence, by part (ii) of Proposition 13, $\{Q_k\}_{k=0}^{\varkappa_V}$ is a V-basis of $\mathcal{P}_N$ selected from $\{P_k\}_{k=0}^{\infty}$. In fact, due to Proposition 13, all V-bases of $\mathcal{P}_N$ selected from $\{P_k\}_{k=0}^{\infty}$ can be obtained this way.

(b) *If $\{P_k\}_{k=0}^{\infty}$ is a basis of $\mathcal{P}_N$ consisting of real column polynomials, $P_0 \in \mathbb{R}$ and $L : \mathcal{P}_N \to \mathbb{C}$ is a linear functional such that $L(P_0) \in \mathbb{R}$ and $L(P_i P_j^*) = 0$ for all $i \neq j$, then L is Hermitian* (cf. Corollary 23, Lemma 29 and Proposition 32 for other criterions for L to be Hermitian). Indeed, taking $p \in \mathcal{P}_N$, we infer from Lemma 3 that there exists a finite sequence of scalar rows $\{D_k\}_{k=0}^{\infty}$ such that $p = \sum_{k=0}^{\infty} D_k P_k$. Since entries of P_k are real, we obtain $p^* = \sum_{k=0}^{\infty} P_k^{\top} D_k^*$. By the quasi-orthogonality assumption, we have $L(P_k) = \frac{1}{P_0^*} L(P_k P_0^*) = 0$ for all $k \geqslant 1$. As a consequence, $L(p) = D_0 L(P_0)$ while $L(p^*) = L(P_0)\overline{D_0}$, which means that $L(p^*) = \overline{L(p)}$.

Example 15. The statement converse to part (b) of Remark 14 is not true. Namely, there exists a Hermitian linear functional L on $\mathcal{P}_N$ for which there is no basis $\{P_k\}_{k=0}^{\infty}$ of $\mathcal{P}_N$ such that $P_0 \in \mathbb{C}$ and $L(P_i P_j^*) = 0$ for all $i \neq j$. The functional L defined by $L(p) = \frac{\mathrm{d}p}{\mathrm{d}x}(0)$ for $p \in \mathcal{P}_1$ has the desired properties. Indeed, if there existed a basis $\{P_k\}_{k=0}^{\infty}$ as above, then we would have $L(P_k) = 0$ for all $k \in \mathbb{N}$ (by the definition of L and the quasi-orthogonality of $\{P_k\}_{k=0}^{\infty}$), which would imply $L = 0$, a contradiction.

Example 15 (as well as the whole Section 5) raises an interesting question when there exists a sequence of column polynomials which is quasi-orthogonal with respect to a given non-zero Hermitian linear functional L and which is a rigid $\mathcal{V}_L$-basis. Our answer to the question generalizes [11, Theorem 3.1.6] (see also [10, Theorem I.3.1] for the single variable case). Given a $*$-ideal V in $\mathcal{P}_N$, we set

$$\Xi_k^V = \begin{bmatrix} \Sigma_0^V \\ \vdots \\ \Sigma_k^V \end{bmatrix}, \quad k \in \overline{0, \varkappa_V}. \tag{29}$$

Proposition 16. *Let $L : \mathcal{P}_N \to \mathbb{C}$ be a non-zero Hermitian linear functional and let $V = \mathcal{V}_L$. Then the following conditions are equivalent*:

(i) *there exists a rigid V-basis $\{Q_k\}_{k=0}^{\varkappa_V}$ of $\mathcal{P}_N$ composed of real column polynomials such that $L(Q_i Q_j^*) = 0$ for all $i \neq j$;*
(ii) *there exists a rigid V-basis $\{Q_k\}_{k=0}^{\varkappa_V}$ of $\mathcal{P}_N$ such that $L(Q_i Q_j^*) = 0$ for all $i \neq j$;*
(iii) *the matrix $L(\Xi_k^V (\Xi_k^V)^*)$ is non-singular for every $k \in \overline{0, \varkappa_V}$;*
(iv) *rank $L(\Xi_k^{\{0\}} (\Xi_k^{\{0\}})^*) = d_V(0) + \cdots + d_V(k)$ for every integer $k \geqslant 0$.*

Proof. The implication (i)⇒(ii) is trivial.

(ii)⇒(iii) Fix $k \in \overline{0, \varkappa_V}$. That the matrix $L(\Xi_k^V (\Xi_k^V)^*) = [L(\Sigma_i^V (\Sigma_j^V)^*)]_{i,j=0}^k$ is non-singular will follow provided we show that the ensuing matrix equation

$$[D_{k,0}, \ldots, D_{k,k}] \begin{bmatrix} L(\Sigma_0^V (\Sigma_0^V)^*) & \ldots & L(\Sigma_0^V (\Sigma_k^V)^*) \\ \vdots & \ddots & \vdots \\ L(\Sigma_k^V (\Sigma_0^V)^*) & \ldots & L(\Sigma_k^V (\Sigma_k^V)^*) \end{bmatrix} = [0, \ldots, 0], \tag{30}$$

has only the zero solution $[D_{k,0}, \ldots, D_{k,k}]$, where $D_{k,j}$ is a scalar $d_V(k) \times d_V(j)$-matrix for $j = 0, \ldots, k$. Set $P_k = D_{k,0}\Sigma_0^V + \cdots + D_{k,k}\Sigma_k^V$. By Lemma 3 and (9), it now suffices to verify that $P_k \overset{V}{=} 0$. It follows from (30) that $L(P_k(\Sigma_j^V)^*) = 0$ for all $j \in \overline{0, k}$. Since $\mathcal{P}_N^{\langle k \rangle}$ is the linear V-span of $\{\Sigma_j^V\}_{j=0}^k$ and V is a $*$-ideal contained in ker L, we see that $L(P_k Q_j^*) = 0$ for all $j \in \overline{0, k}$. However $L(P_k Q_j^*) = 0$ for all $j > k$ (because $\mathcal{P}_N^{\langle k \rangle}$ is a linear V-span of $\{Q_j\}_{j=0}^k$, $V \subseteq$ ker L and $L(Q_i Q_j^*) = 0$ for all $i \neq j$). Employing the fact that $\{Q_j\}_{j=0}^{\varkappa_V}$ is a V-basis of $\mathcal{P}_N$ as well as the equality $\mathcal{V}_L = V$, we conclude that $P_k \overset{V}{=} 0$.

(iii)⇒(i) Since L is Hermitian, we have $L(\Xi_k^V (\Xi_k^V)^*)^* = L(\Xi_k^V (\Xi_k^V)^*) = L(\Xi_k^V (\Xi_k^V)^*)^{\mathsf{T}}$. Hence the matrix $L(\Xi_k^V (\Xi_k^V)^*)$ is real. By the non-singularity of $L(\Xi_k^V (\Xi_k^V)^*)$, for each $k \in \overline{1, \varkappa_V}$, there exists a unique system of real scalar matrices $D_{k,0}, \ldots, D_{k,k}$ such that

$$\begin{bmatrix} L(\Sigma_0^V (\Sigma_0^V)^*) & \ldots & L(\Sigma_0^V (\Sigma_{k-1}^V)^*) & L(\Sigma_0^V (\Sigma_k^V)^*) \\ \vdots & \ddots & \vdots & \vdots \\ L(\Sigma_{k-1}^V (\Sigma_0^V)^*) & \ldots & L(\Sigma_{k-1}^V (\Sigma_{k-1}^V)^*) & L(\Sigma_{k-1}^V (\Sigma_k^V)^*) \\ L(\Sigma_k^V (\Sigma_0^V)^*) & \ldots & L(\Sigma_k^V (\Sigma_{k-1}^V)^*) & L(\Sigma_k^V (\Sigma_k^V)^*) \end{bmatrix} \begin{bmatrix} D_{k,0}^{\mathsf{T}} \\ \vdots \\ D_{k,k-1}^{\mathsf{T}} \\ D_{k,k}^{\mathsf{T}} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix}, \tag{31}$$

where the matrix $D_{k,j}$ is of size $d_V(k) \times d_V(j)$ and I stands for the identity matrix of size $d_V(k) \times d_V(k)$. For $k \in \overline{0, \varkappa_V}$, we define the real column polynomial $Q_k = D_{k,0}\Sigma_0^V + \cdots + D_{k,k}\Sigma_k^V$ with $D_{0,0} \stackrel{\text{df}}{=} 1$. Since $D_{k,i}^* = D_{k,i}^\top$, we infer from (31) that $L(\Sigma_j^V Q_k^*) = 0$ for all $j \in \overline{0, k-1}$, which implies $L(Q_j Q_k^*) = 0$ for all $j \in \overline{0, k-1}$. Taking transpose and using $Q_j^* = Q_j^\top$, we get $L(Q_k Q_j^*) = 0$ for all $j \in \overline{0, k-1}$. Summarizing, we have proved that $L(Q_i Q_j^*) = 0$ for all $i \neq j$. It follows from the definition that $\deg Q_k \leqslant k$. Owing to Proposition 9, it suffices to show that the matrix $D_{k,k}$ is non-singular. Since $D_{0,0} = 1$, we can assume that $k > 0$. If a is a scalar column of length $d_V(k)$ and $D_{k,k}^\top a = 0$, then, by (31), we have

$$\begin{bmatrix} L(\Sigma_0^V(\Sigma_0^V)^*) & \dots & L(\Sigma_0^V(\Sigma_{k-1}^V)^*) \\ \vdots & \ddots & \vdots \\ L(\Sigma_{k-1}^V(\Sigma_0^V)^*) & \dots & L(\Sigma_{k-1}^V(\Sigma_{k-1}^V)^*) \end{bmatrix} \begin{bmatrix} D_{k,0}^\top a \\ \vdots \\ D_{k,k-1}^\top a \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

However, the matrix $L(\Xi_{k-1}^V(\Xi_{k-1}^V)^*)$ is non-singular, and hence $D_{k,0}^\top a = 0, \ldots, D_{k,k-1}^\top a = 0$. This, when combined with (31), leads to $a = 0$.

(iii)⇔(iv) Suppose that $\varkappa_V = \infty$ and fix $k \in \mathbb{N}$. Since $\{\Sigma_j^V\}_{j=0}^\infty$ is a rigid V-basis of $\mathcal{P}_N$, there exists a scalar matrix E_k such that $\Xi_k^{\{0\}} \stackrel{V}{=} E_k \Xi_k^V$. This and the fact that $L(\Xi_k^V(\Xi_k^V)^*)$ is a submatrix of $L(\Xi_k^{\{0\}}(\Xi_k^{\{0\}})^*)$ give us

$$\begin{aligned} \operatorname{rank} L(\Xi_k^V(\Xi_k^V)^*) &\leqslant \operatorname{rank} L(\Xi_k^{\{0\}}(\Xi_k^{\{0\}})^*) \\ &= \operatorname{rank} E_k L(\Xi_k^V(\Xi_k^V)^*) E_k^* \leqslant \operatorname{rank} L(\Xi_k^V(\Xi_k^V)^*). \end{aligned}$$

Since $L(\Xi_k^V(\Xi_k^V)^*)$ is a square matrix of dimension $d_V(0) + \cdots + d_V(k)$, the equivalence of (iii) and (iv) follows.

If $\varkappa_V < \infty$ and $k > \varkappa_V$, then a similar reasoning shows that $\Xi_k^{\{0\}} \stackrel{V}{=} E_k \Xi_{\varkappa_V}^V$ with an appropriate scalar matrix E_k, and in consequence

$$\operatorname{rank} L(\Xi_k^{\{0\}}(\Xi_k^{\{0\}})^*) = \operatorname{rank} L(\Xi_{\varkappa_V}^V(\Xi_{\varkappa_V}^V)^*),$$

which together with $d_V(l) = 0$ for $l > \varkappa_V$ completes the proof. □

Remark 17. Notice that if we drop the assumption that L is Hermitian supposing instead that $V = \mathcal{V}_L$ is a $*$-ideal, then conditions (i), (ii) and (iii) of Proposition 16 remain equivalent provided (iii) is strengthened by $L(\Xi_k^V(\Xi_k^V)^*) = L(\Xi_k^V(\Xi_k^V)^*)^*$ for $k \in \overline{0, \varkappa_V}$. Indeed, the proof of (ii)⇒(iii) works under the assumption $V = V^*$, while the proof of (iii)⇒(i) remains valid provided $L(\Xi_k^V(\Xi_k^V)^*) = L(\Xi_k^V(\Xi_k^V)^*)^*$ for $k \in \overline{0, \varkappa_V}$.

One may obtain yet another version of Proposition 16 if L is assumed to be an arbitrary linear functional with $\mathcal{V}_L \neq \mathcal{P}_N$. Then conditions (ii), (iii) and (iv) are still equivalent whenever taking adjoint (*) is replaced by transposing ($^\top$).

We are now in a position to formulate a version of Favard's theorem for quasi-orthogonality of polynomials of several variables with respect to a Hermitian linear functional on $\mathcal{P}_N$.

Theorem 18. *Let V be a proper $*$-ideal in $\mathcal{P}_N$, $L : \mathcal{P}_N \to \mathbb{C}$ be a linear functional and $\{Q_k\}_{k=0}^n$ $(0 \leqslant n \leqslant \infty)$ be a sequence of real column polynomials such that $Q_0 \neq 0$. Consider the following two conditions*:

(A) $\{Q_k\}_{k=0}^n$ *is a rigid V-basis of* $\mathcal{P}_N$, $L(Q_i Q_j^*) = 0$ *for all* $i \neq j$, *and* $V = \mathcal{V}_L$;

(B) $n = \varkappa_V$ *and there exists a system* $\{[A_{k,j}, B_{k,j}, C_{k,j}]\}_{k=0}^{\varkappa_V}{}_{j=1}^{N}$ *of scalar matrices such that*

- (B-i) $X_j Q_k \overset{V}{=} A_{k,j} Q_{k+1} + B_{k,j} Q_k + C_{k,j} Q_{k-1}$ *for all* $j \in \overline{1, N}$ *and* $k \in \overline{0, \varkappa_V}$, *where* $C_{0,j} \overset{\text{df}}{=} 1$ *and* $Q_{-1} \overset{\text{df}}{=} 0$; *if* $\varkappa_V < \infty$, *then* $A_{\varkappa_V, j} \overset{\text{df}}{=} [1, \ldots, 1]^*$ *with* $\ell(A_{\varkappa_V, j}) = \ell(Q_{\varkappa_V})$ *and* $Q_{\varkappa_V+1} \overset{\text{df}}{=} 0$,
- (B-ii) *the length of* Q_k *is less than or equal to* $d_V(k)$ *for every* $k \in \overline{0, \varkappa_V}$,
- (B-iii) $\deg Q_k \leqslant k$ *for every* $k \in \overline{0, \varkappa_V}$,
- (B-iv) *the matrix* $[C_{k,1}, \ldots, C_{k,N}]$ *is of maximal rank for every* $k \in \overline{0, \varkappa_V}$.

Then

(a) (B-i), (B-ii) *and* (B-iii) *imply that the matrix* $[A_{k,1}^*, \ldots, A_{k,N}^*]^*$ *is injective for every* $k \in \overline{0, \varkappa_V}$, $\{Q_j\}_{j=0}^n$ *is a rigid V-basis of* $\mathcal{P}_N$, *the linear functional L defined by*

$$L|_V = 0, L(Q_0) = Q_0 \quad \textit{and} \quad L(Q_k) = 0 \quad \textit{for all } k \in \overline{1, \varkappa_V}, \tag{32}$$

is Hermitian and $L(Q_k Q_l^*) = 0$ *for all* $k \neq l$,

(b) (B) *implies* (A) *and the non-singularity of* $L(Q_k Q_k^*)$ *for every* $k \in \overline{0, \varkappa_V}$, *where L is defined by* (32),

(c) (A) *implies* (B); *moreover, if a linear functional* $L' : \mathcal{P}_N \to \mathbb{C}$ *satisfies* (A), *then* $L'(X^0) \neq 0$ *and the functional* $\frac{1}{L'(X^0)} L'$ *fulfills* (32).

Regarding Theorem 18, the reader should address himself to Example 55 which reveals the importance of rigidity in (A). In turn, Proposition 21 provides equivalent forms of condition (A). The matrix appearing in (B-iv) is in fact surjective.

Proof of Theorem 18. As the reader can easily check, the assumption that all the column polynomials Q_k are real is only used in Step 3 below in order to prove that the functional L defined by (32) is Hermitian. Once the Hermitian property of L is established, the subsequent parts of the proof do not explicitly refer to $\{Q_k\}_{k=0}^n$ being real column polynomials. Parts (a) and (b) of the conclusion are shown in the unit (B)⇒(A) below, while the rest of the proof is contained in the unit (A)⇒(B).

(B)⇒(A) We split the proof into a few steps, starting with a result whose generality is surplus to requirements.

Step 1: If $\{Q_k\}_{k=0}^{\varkappa_V}$ is a sequence of column polynomials satisfying (B-i), $Q_0 \in \mathbb{C} \setminus \{0\}$ and for every $k \in \overline{0, \varkappa_V}$, there exists a scalar matrix G_k and a column polynomial R_k such that

$$Q_k \overset{V}{=} G_k \Sigma_k^V + R_k \quad \text{and} \quad \deg R_k < k, \tag{33$_k$}$$

then for every $k \in \overline{0, \varkappa_V}$, the matrix G_k is injective, $d_V(k) \leqslant \ell(Q_k)$ and $\Pi_V(\mathcal{P}_N^{\langle k]}) = \operatorname{lin} \Pi_V(\bigcup_{i=0}^k Q_i)$.

The proof is by induction on k. The case $k = 0$ is obvious. Assume that the conclusion of Step 1 is true for a fixed integer $0 \leqslant k < \varkappa_V$. As in the proof of Proposition 13, we rewrite condition (B-i) in the column form (25) and then, applying $(33)_{k-1}$, $(33)_k$, $(33)_{k+1}$ and Lemma 5, we compare the $(k+1)$th coefficients of column polynomials (relative to V) appearing on both sides of (25). In consequence, we obtain (see (26) and (27))

$$\begin{bmatrix} G_k & 0 & \cdots & 0 \\ 0 & G_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_k \end{bmatrix} M_{k+1} = \begin{bmatrix} A_{k,1} \\ A_{k,2} \\ \vdots \\ A_{k,N} \end{bmatrix} G_{k+1}. \tag{34}$$

The injectivity of G_k and M_{k+1} implies via (34) the injectivity of G_{k+1}. Since, by $(33)_{k+1}$, the matrix G_{k+1} has $d_V(k+1)$ columns and $\ell(Q_{k+1})$ rows, we conclude that $d_V(k+1) \leqslant \ell(Q_{k+1})$. Multiplying both sides of $Q_{k+1} \overset{V}{=} G_{k+1}\Sigma_{k+1}^V + R_{k+1}$ by the left inverse of G_{k+1} and using the induction hypothesis, we see that $\Pi_V(\Sigma_{k+1}^V) \subseteq \operatorname{lin} \Pi_V(\bigcup_{i=0}^{k+1} Q_i)$, which by (9) implies $\Pi_V(\mathcal{P}_N^{\langle k+1]}) \subseteq \operatorname{lin} \Pi_V(\bigcup_{i=0}^{k+1} Q_i)$. The reverse inclusion is obvious due to $(33)_{k+1}$. This completes the induction argument.

In Steps 2–5 the sequence $\{Q_k\}_{k=0}^{\varkappa_V}$ is supposed to satisfy the assumptions of Theorem 18 as well as conditions (B-i), (B-ii) and (B-iii).

Step 2: $\{Q_k\}_{k=0}^{\varkappa_V}$ is a rigid V-basis of $\mathcal{P}_N$ and for every $k \in \overline{0, \varkappa_V}$, the matrix $[A_{k,1}^*, \ldots, A_{k,N}^*]^*$ is injective, hence of maximal rank.

Indeed, since $\{\Sigma_k^V\}_{k=0}^{\varkappa_V}$ is a rigid V-basis of $\mathcal{P}_N$ and $\deg Q_k \leqslant k$ for every $k \in \overline{0, \varkappa_V}$, Lemma 3 implies that $\{Q_k\}_{k=0}^{\varkappa_V}$ satisfies the assumptions of Step 1. Hence, by (B-ii) and Step 1, for every $k \in \overline{0, \varkappa_V}$, $d_V(k) = \ell(Q_k)$ and G_k is an injective square matrix. This, when combined with Proposition 9, shows that $\{Q_k\}_{k=0}^{\varkappa_V}$ is a rigid V-basis of $\mathcal{P}_N$. In virtue of (34), the matrix $[A_{k,1}^*, \ldots, A_{k,N}^*]^*$ is injective for every integer $0 \leqslant k < \varkappa_V$; the case $k = \varkappa_V < \infty$ is trivial.

Step 3: The functional L given by (32) is a well defined Hermitian linear functional such that $L(Q_i Q_j^*) = 0$ for all $i \neq j$.

Since, by Step 2, $\{Q_k\}_{k=0}^{\varkappa_V}$ is a V-basis of $\mathcal{P}_N$, we see that $\bigcup_{i=0}^{\varkappa_V} Q_i$ is a basis of $\operatorname{lin} \bigcup_{i=0}^{\varkappa_V} Q_i$ and $\mathcal{P}_N$ is the direct sum of V and $\operatorname{lin} \bigcup_{i=0}^{\varkappa_V} Q_i$. This justifies the correctness of the definition and the uniqueness of a linear functional L satisfying (32). In fact, the functional L is of the form $L(p) \overset{\text{df}}{=} L_0(p + V)$ for $p \in \mathcal{P}_N$, where $L_0 : \mathcal{P}_N / V \to \mathbb{C}$ is the unique linear functional defined by $L_0(Q_0 + V) = Q_0$ and $L_0(q + V) = 0$ for $q \in \bigcup_{k=1}^{\varkappa_V} Q_k$. It is obvious that $V \subseteq \ker L$. We show that L is Hermitian. Indeed, if $p \in \mathcal{P}_N$, then there exist $p_1 \in V$ and $p_2 \in \operatorname{lin} \bigcup_{i=0}^{\varkappa_V} Q_i$ such that $p = p_1 + p_2$. Clearly, $L(p_1^*) = 0$ because V is a $*$-ideal. Since $\bigcup_{i=0}^{\varkappa_V} Q_i$ is composed of real polynomials, one can check that $L(p_2^*) = \overline{L(p_2)}$. Hence $L(p^*) = \overline{L(p)}$.

We now show that for every $k \in \overline{0, \varkappa_V}$

$$L(Q_i Q_j^*) = 0, \quad 0 \leqslant j \leqslant k, \ j < i \leqslant \varkappa_V. \tag{35}$$

The proof is by induction on k. By (32), the case $k = 0$ is obvious. Assume that (35) is true for a fixed integer $0 \leqslant k < \varkappa_V$. Let i be an integer such that $k + 1 < i \leqslant \varkappa_V$. By (B-i), we have

$$(X_n Q_i) Q_k^* \overset{V}{=} (A_{i,n} Q_{i+1} + B_{i,n} Q_i + C_{i,n} Q_{i-1}) Q_k^*$$

for all $n = 1, \ldots, N$. This and the induction hypothesis imply

$$L(X_n Q_i Q_k^*) = 0, \quad n = 1, \ldots, N. \tag{36}$$

Multiplying both sides of (25) by the left inverse of $[A_{k,1}^*, \ldots, A_{k,N}^*]^*$ (which exists due to Step 2), we get scalar matrices $\{D_n\}_{n=1}^N$ and $\{E_l\}_{l=0}^k$ such that

$$Q_{k+1} \overset{V}{=} \sum_{n=1}^{N} X_n D_n Q_k + \sum_{l=0}^{k} E_l Q_l. \tag{37}$$

This, the assumption that V is a $*$-ideal, the induction hypothesis and (36) give us $L(Q_i Q_{k+1}^*) = 0$, which completes the induction argument. Since the functional L is Hermitian, we deduce from (35) that $L(Q_i Q_j^*) = 0$ for all $i \neq j$.

In the last two steps of the proof we assume that condition (B-iv) holds.

Step 4: $L(Q_k Q_k^*)$ is non-singular for every $k \in \overline{0, \varkappa_V}$.

We proceed by induction on k. The case $k = 0$ is clear. Assume that $L(Q_k Q_k^*)$ is non-singular for a fixed integer $0 \leqslant k < \varkappa_V$. Arguing as in the last paragraph of the proof of Proposition 13 and applying (B-i) and Step 3, we obtain the equality (28). In virtue of (B-i) and Step 2, the matrix $[C_{k+1,1}, \ldots, C_{k+1,N}]$ has $d_V(k + 1)$ rows and $N d_V(k)$ columns. By (8) and (B-iv), the matrix $[C_{k+1,1}, \ldots, C_{k+1,N}]$ is surjective, or equivalently its adjoint $[C_{k+1,1}, \ldots, C_{k+1,N}]^*$ is injective. Hence, by the induction hypothesis and (28), $L(Q_{k+1} Q_{k+1}^*)$ is an injective square matrix.

Step 5: $V = \mathcal{V}_L$.

Let p be in $\mathcal{V}_L$. By Step 2 and Lemma 3, there exists a finite sequence $\{D_j\}_{j=0}^{\varkappa_V}$ of scalar rows such that $p \overset{V}{=} \sum_{j=0}^{\varkappa_V} D_j Q_j$. Taking $i \in \overline{0, \varkappa_V}$, we infer from Step 3 that $D_i L(Q_i Q_i^*) = L(p Q_i^*) = 0$. Hence, by Step 4, $D_i = 0$. Consequently, p is in V, which shows that $\mathcal{V}_L \subseteq V$. The converse inclusion $V \subseteq \mathcal{V}_L$ follows from $V \subseteq \ker L$.

(A)⇒(B) According to Proposition 10, the sequence $\{P_k\}_{k=0}^\infty$ defined by (16) is a rigid basis of $\mathcal{P}_N$, $\{Q_k\}_{k=0}^{\varkappa_V} \preccurlyeq \{P_k\}_{k=0}^\infty$ and $\bigcup_{k=0}^\infty P_k \setminus \bigcup_{k=0}^{\varkappa_V} Q_k \subseteq V$. As a consequence, $L(P_k P_j^*) = 0$ for all $k \neq j$. Hence Proposition 13 completes the proof of (A)⇒(B).

Finally, if $L' : \mathcal{P}_N \to \mathbb{C}$ is a linear functional satisfying (A), then $L'|_V = 0$ (because $\mathcal{V}_{L'} \subseteq \ker L'$) and $L'(Q_k) = \frac{1}{Q_0^*} L'(Q_k Q_0^*) = 0$ for every $k \in \overline{1, \varkappa_V}$, which implies $L' = L'(X^0) L$ with L as in (32) (recall that $\{Q_k\}_{k=0}^{\varkappa_V}$ is a V-basis of $\mathcal{P}_N$). This yields $L'(X^0) \neq 0$, because otherwise $L' = 0$, and so $V = \mathcal{V}_{L'} = \mathcal{P}_N$. □

Remark 19. Suppose the assumptions of Theorem 18 are satisfied. If conditions (B-i), (B-ii) and (B-iii) are fulfilled and L is given by (32), then

(a) *all scalar matrices* $A_{k,j}, B_{k,j}, C_{k,j}$ *appearing in* (B-i) *are real*. Indeed, taking adjoints of both sides of the relation "$\overset{V}{=}$" in (B-i), then transposing them and finally exploiting the fact that all Q_k are real column polynomials, we deduce that (B-i) holds for matrices $A_{k,j}^{*\mathsf{T}}$, $B_{k,j}^{*\mathsf{T}}$, $C_{k,j}^{*\mathsf{T}}$ instead of $A_{k,j}$, $B_{k,j}$, $C_{k,j}$, respectively. Since $\{Q_k\}_{k=0}^{\varkappa_V}$ is a V-basis of $\mathcal{P}_N$, we conclude that $A_{k,j}^{*\mathsf{T}} = A_{k,j}$, $B_{k,j}^{*\mathsf{T}} = B_{k,j}$ and $C_{k,j}^{*\mathsf{T}} = C_{k,j}$, which proves our claim.

(b) *$B_{k,j} L(Q_k Q_k^*) = L(Q_k Q_k^*) B_{k,j}^*$ for all $k \in \overline{0, \varkappa_V}$ and $j \in \overline{1, N}$.* Indeed, applying (B-i) to the equality $(X_j Q_k) Q_k^* = Q_k (X_j Q_k)^*$, then letting the functional L act on both sides of it and simultaneously using the inclusion $V \subseteq \ker L$ and Step 3, we get (b).

(c) *$V = \mathcal{V}_L$ if and only if $L(Q_k Q_k^*)$ is non-singular for every $k \in \overline{0, \varkappa_V}$.*

This follows from part (a) of Theorem 18 and Proposition 21.

Example 20. Implication (B)⇒(A) in Theorem 18 is no longer true if we drop assumption (B-iv). To see this put $N = 1$, $V = \{0\}$ and $Q_k = X^k$ for $k \in \mathbb{N}$. It is clear that the sequence $\{Q_k\}_{k=0}^{\infty}$ satisfies (B-i), (B-ii) and (B-iii) with $A_{k,1} \overset{\mathrm{df}}{=} 1$, $B_{k,1} \overset{\mathrm{df}}{=} 0$, $C_{0,1} \overset{\mathrm{df}}{=} 1$ and $C_{k+1,1} \overset{\mathrm{df}}{=} 0$ for $k \in \mathbb{N}$. However, $\{Q_k\}_{k=0}^{\infty}$ fails to satisfy (B-iv). The functional L defined by (32) is of the form $L(p) = p(0)$ for $p \in \mathcal{P}_1$. As a consequence, L is Hermitian (in fact it is positive definite, cf. Section 8) and $\mathcal{V}_L = \ker L$. Clearly, $V \subsetneq \mathcal{V}_L$ and $L(Q_k Q_k^*) = 0$ for all integers $k \geqslant 1$.

6. Quasi-orthogonality: degree versus rank

Let us begin by formulating some equivalent forms of the equality $V = \mathcal{V}_L$ which is a mysterious part of condition (A) in Theorem 18.

Proposition 21. *Let V be a proper $*$-ideal in $\mathcal{P}_N$, $L : \mathcal{P}_N \to \mathbb{C}$ be a linear functional such that $V \subseteq \ker L$, and $\{Q_k\}_{k=0}^{n}$ $(0 \leqslant n \leqslant \infty)$ be a V-basis of $\mathcal{P}_N$ such that $L(Q_i Q_j^*) = 0$ for all $i \neq j$. Then the following conditions are equivalent*:

(i) *$V = \mathcal{V}_L$,*
(ii) *$L(Q_k Q_k^*)$ is non-singular for every $k \in \overline{0, n}$,*
(iii) *the sequence $\{Q_k\}_{k=0}^{n}$ is linearly $\mathcal{V}_L$-independent.*

Moreover, if $L(Q_k Q_k^*) = L(Q_k Q_k^*)^*$ for every $k \in \overline{0, n}$, then (i) *is equivalent to*

(iv) *there exists a V-basis $\{P_k\}_{k=0}^{n}$ of $\mathcal{P}_N$ such that $L(P_i P_j^*) = 0$ for all $i \neq j$ and $L(P_k P_k^*)$ is a non-singular diagonal real matrix for every $k \in \overline{0, n}$.*

If $\{Q_k\}_{k=0}^{n}$ is rigid (*resp. composed of real column polynomials*), *then $\{P_k\}_{k=0}^{n}$ in* (iv) *can be chosen to be a rigid V-basis of $\mathcal{P}_N$* (*resp. to consist of real column polynomials*).

Proof. (i)⇒(ii) Suppose that, contrary to our claim, $L(Q_k Q_k^*)$ is singular for some $k \in \overline{0, n}$. Then there exists a scalar row $D \neq 0$ such that $0 = DL(Q_k Q_k^*) = L((DQ_k) Q_k^*)$. By the quasi-orthogonality assumption, $L((DQ_k) Q_i^*) = 0$ for every $i \in \overline{0, n}$. Since $\{Q_i\}_{i=0}^{n}$ is a V-basis of $\mathcal{P}_N$ and $V \subseteq \ker L$, we conclude that $L((DQ_k) q^*) = 0$ for all $q \in \mathcal{P}_N$, which means that $DQ_k \in \mathcal{V}_L = V$ (compare with the proof of (21)). This contradicts the linear V-independence of $\{Q_i\}_{i=0}^{n}$.

(ii)⇒(iii) If $\sum_{k=0}^{n} D_k Q_k \overset{\mathcal{V}_L}{=} 0$, where $\{D_k\}_{k=0}^n$ is a finite sequence of scalar rows, then by the quasi-orthogonality assumption and $\mathcal{V}_L \subseteq \ker L$ we obtain

$$D_j L(Q_j Q_j^*) = L\left(\sum_{k=0}^{n} D_k Q_k Q_j^*\right) = 0, \quad j \in \overline{0, n}.$$

This and the non-singularity of the matrix $L(Q_j Q_j^*)$ imply that $D_j = 0$ for all $j \in \overline{0, n}$, which gives us (iii).

(iii)⇒(i) Since $V \subseteq \mathcal{V}_L$, the equality $V = \mathcal{V}_L$ is a consequence of the following fact: for any two proper ideals $V_1 \subseteq V_2 \subseteq \mathcal{P}_N$, if there exists a linearly V_2-independent set $B \subseteq \mathcal{P}_N$ which is simultaneously a V_1-basis of $\mathcal{P}_N$, then $V_1 = V_2$.

(iv)⇒(i) This is a direct consequence of (ii)⇒(i) applied to $\{P_k\}_{k=0}^n$.

Assume now that the matrix $L(Q_k Q_k^*)$ is symmetric for every $k \in \overline{0, n}$.

(i)⇒(iv) Fix $k \in \overline{0, n}$. By (i)⇒(ii), the matrix $L(Q_k Q_k^*)$ is non-singular and symmetric. Hence, there exists a unitary (scalar) matrix U_k such that the matrix $U_k L(Q_k Q_k^*) U_k^*$ is non-singular and diagonal. Set $P_k = U_k Q_k$. It is now easily seen that $\{P_k\}_{k=0}^n$ is the desired V-basis of $\mathcal{P}_N$. If the V-basis $\{Q_k\}_{k=0}^n$ is rigid, then by Proposition 9 so is $\{P_k\}_{k=0}^n$. If the column polynomial Q_k is real, then the matrix $L(Q_k Q_k^{\mathsf{T}})$ is real and symmetric. Thus, the matrix U_k can be chosen to be real. Consequently, the column P_k is real as well. □

Corollary 22. *Under the assumptions of Theorem* 18, (A) *is equivalent to*

(†) *$\{Q_k\}_{k=0}^n$ is a rigid V-basis of $\mathcal{P}_N$, $L(Q_i Q_j^*) = 0$ for all $i \neq j$, $L(Q_k Q_k^*)$ is non-singular for every $k \in \overline{0, n}$ and $V \subseteq \ker L$.*

If moreover $L(X^0) \in \mathbb{R}$, then (A) *is equivalent to*

(††) *$\{Q_k\}_{k=0}^n$ is a rigid V-basis of $\mathcal{P}_N$, $L(Q_i Q_j^*) = 0$ for all $i \neq j$, $V \subseteq \ker L$ and there exists a rigid V-basis $\{P_k\}_{k=0}^n$ of $\mathcal{P}_N$ composed of real column polynomials such that $L(P_i P_j^*) = 0$ for all $i \neq j$ and $L(P_k P_k^*)$ is a non-singular diagonal real matrix for every $k \in \overline{0, n}$.*

Proof. It is sufficient to apply Proposition 21. The proof of (A)⇒(††) requires the symmetry of $L(Q_k Q_k^*)$ which follows from the Hermitian property of L guaranteed by Theorem 18. □

Corollary 23. *Let V be a proper $*$-ideal in $\mathcal{P}_N$, $L : \mathcal{P}_N \to \mathbb{C}$ be a linear functional such that $V \subseteq \ker L$, and $\{Q_k\}_{k=0}^n$ $(0 \leqslant n \leqslant \infty)$ be a V-basis of $\mathcal{P}_N$ such that $L(Q_i Q_j^*) = 0$ for all $i \neq j$ and $L(Q_k Q_k^*)$ is a non-singular symmetric matrix for every $k \in \overline{0, n}$. Then the functional L is Hermitian.*

Proof. By (ii)⇒(iv) of Proposition 21, there is no loss of generality in assuming that $L(Q_k Q_k^*)$ is a non-singular diagonal real matrix for every $k \in \overline{0, n}$. Arrange members of the set $\bigcup_{k=0}^{n} Q_k$ in a sequence $\{q_j\}_{j=1}^{s}$ so that $q_k \neq q_l$ for all $k \neq l$. Then $\{q_j\}_{j=1}^{s}$ is a V-basis of $\mathcal{P}_N$ such that

$$L(q_k q_l^*) = 0 \quad \text{for all } k \neq l \quad \text{and} \quad L(q_j q_j^*) \in \mathbb{R} \setminus \{0\} \quad \text{for every } j \in \overline{1, n}. \tag{38}$$

We show that an L with these properties must be Hermitian. Notice first that

$$L(pp^*) \in \mathbb{R} \quad \text{for all } p \in \mathcal{P}_N. \tag{39}$$

Indeed, since $\{q_j\}_{j=1}^{s}$ is a V-basis of $\mathcal{P}_N$, there exists a finite system $\{\alpha_j\}_{j=1}^{s}$ of complex numbers such that $p \overset{V}{=} \sum_{j=1}^{s} \alpha_j q_j$. This, the assumption that V is a $*$-ideal contained in ker L and (38) yield

$$L(pp^*) = L\left(\left(\sum_{i=1}^{s} \alpha_i q_i\right)\left(\sum_{j=1}^{s} \bar{\alpha}_j q_j^*\right)\right) = \sum_{i=1}^{s} |\alpha_i|^2 L(q_i q_i^*) \in \mathbb{R}.$$

We now turn to the final stage of our proof. Take $p \in \mathcal{P}_N$. Then, by (39), we have

$$\alpha L(p) + \bar{\alpha} L(p^*) = L((\bar{\alpha} + p)(\bar{\alpha} + p)^*) - |\alpha|^2 L(X^0(X^0)^*) - L(pp^*) \in \mathbb{R},$$
$$\alpha \in \mathbb{C}.$$

Substituting $\alpha = 1, \mathrm{i}$, we get $L(p^*) = \overline{L(p)}$. This completes the proof. □

Remark 24. Let V be a proper $*$-ideal in $\mathcal{P}_N$, $L, \widetilde{L} : \mathcal{P}_N \to \mathbb{C}$ be linear functionals and $\{Q_k\}_{k=0}^{\varkappa_V}$, $\{\widetilde{Q}_k\}_{k=0}^{\varkappa_V}$ be sequences of real column polynomials such that $Q_k \overset{V}{=} \widetilde{Q}_k$ for all $k \in \overline{0, \varkappa_V}$. Assume that the triplet $(V, L, \{Q_k\}_{k=0}^{\varkappa_V})$ satisfies (A) (cf. Theorem 18). Then $(V, \widetilde{L}, \{\widetilde{Q}_k\}_{k=0}^{\varkappa_V})$ satisfies (A) if and only if $\widetilde{L}$ is a non-zero scalar multiple of L and $\deg(\widetilde{Q}_k - Q_k) \leqslant k$ for all $k \in \overline{0, \varkappa_V}$ (it may happen that $\deg(\widetilde{Q}_k - Q_k) = k$, see Lemma 25). Moreover, if $(V, \widetilde{L}, \{\widetilde{Q}_k\}_{k=0}^{\varkappa_V})$ satisfies (A), then $A_{k,j} = \widetilde{A}_{k,j}$, $B_{k,j} = \widetilde{B}_{k,j}$ and $C_{k,j} = \widetilde{C}_{k,j}$ for all k, j, where $A_{k,j}$, $B_{k,j}$ and $C_{k,j}$ (resp. $\widetilde{A}_{k,j}$, $\widetilde{B}_{k,j}$ and $\widetilde{C}_{k,j}$) are scalar matrices attached to $(V, L, \{Q_k\}_{k=0}^{\varkappa_V})$ (resp. $(V, \widetilde{L}, \{\widetilde{Q}_k\}_{k=0}^{\varkappa_V})$) via implication (A)⇒(B) of Theorem 18.

Indeed, if $(V, \widetilde{L}, \{\widetilde{Q}_k\}_{k=0}^{\varkappa_V})$ satisfies (A), then evidently $\deg(\widetilde{Q}_k - Q_k) \leqslant k$ for all $k \in \overline{0, \varkappa_V}$. As $\widetilde{Q}_i \widetilde{Q}_j^* \overset{V}{=} Q_i Q_j^*$ and $V = \mathcal{V}_{\widetilde{L}} \subseteq \ker \widetilde{L}$, we get $\widetilde{L}(Q_i Q_j^*) = \widetilde{L}(\widetilde{Q}_i \widetilde{Q}_j^*) = 0$ for all $i \neq j$, which means that the triplets $(V, L, \{Q_k\}_{k=0}^{\varkappa_V})$ and $(V, \widetilde{L}, \{Q_k\}_{k=0}^{\varkappa_V})$ satisfy (A). By part (c) of Theorem 18, $\widetilde{L}$ is a non-zero scalar multiple of L. Since

$$\begin{aligned}\widetilde{A}_{k,j} \widetilde{Q}_{k+1} + \widetilde{B}_{k,j} \widetilde{Q}_k + \widetilde{C}_{k,j} \widetilde{Q}_{k-1} &\overset{V}{=} X_j \widetilde{Q}_k \overset{V}{=} X_j Q_k \\ &\overset{V}{=} A_{k,j} Q_{k+1} + B_{k,j} Q_k + C_{k,j} Q_{k-1} \\ &\overset{V}{=} A_{k,j} \widetilde{Q}_{k+1} + B_{k,j} \widetilde{Q}_k + C_{k,j} \widetilde{Q}_{k-1}\end{aligned}$$

and $\{\widetilde{Q}_k\}_{k=0}^{\varkappa_V}$ is a V-basis of $\mathcal{P}_N$, Lemma 3 implies that $A_{k,j} = \widetilde{A}_{k,j}$, $B_{k,j} = \widetilde{B}_{k,j}$ and $C_{k,j} = \widetilde{C}_{k,j}$ for all k, j.

Suppose now that $\widetilde{L}$ is a non-zero scalar multiple of L and $\deg(\widetilde{Q}_k - Q_k) \leqslant k$ for all $k \in \overline{0, \varkappa_V}$. Then evidently $\deg \widetilde{Q}_k \leqslant k$ for all $k \in \overline{0, \varkappa_V}$. It follows from $\widetilde{Q}_k \overset{V}{=} Q_k$ that $\{\widetilde{Q}_k\}_{k=0}^{\varkappa_V}$ is a V-basis of $\mathcal{P}_N$ (use Lemma 3) and $\ell(\widetilde{Q}_i) = \ell(Q_i) = d_V(i)$ for all $i \in \overline{0, \varkappa_V}$. By

Proposition 6, $\{\widetilde{Q}_k\}_{k=0}^{\varkappa_V}$ is a rigid V-basis of $\mathcal{P}_N$. It is clear that $\widetilde{L}(\widetilde{Q}_i \widetilde{Q}_j^*) = L(Q_i Q_j^*) = 0$ for all $i \neq j$. Thus $(V, \widetilde{L}, \{\widetilde{Q}_k\}_{k=0}^{\varkappa_V})$ satisfies (A).

Let us now discuss the role played by condition (B-iii) in Theorem 18. Given a proper ideal V in $\mathcal{P}_N$ and a column polynomial Q, we define

$$\deg_V Q = \min\{\deg P : P \text{ is a column polynomial such that } Q \overset{V}{=} P\},$$
$$\mathfrak{n}_V = \begin{cases} \min\{\deg p : p \in V \setminus \{0\}\} & \text{if } V \neq \{0\}, \\ \infty & \text{if } V = \{0\}. \end{cases}$$

It is clear that $\deg_V Q \leqslant \deg Q$, $\deg_{\{0\}} Q = \deg Q$ and $\mathfrak{n}_V \geqslant 1$ (because $V \neq \mathcal{P}_N$). Notice that in general there are no relations between $\varkappa_V$ and $\mathfrak{n}_V$ (e.g. if $V = (X_1, X_2) \subseteq \mathcal{P}_2$, then $\varkappa_V = 0 < 1 = \mathfrak{n}_V$, while if $V = (X_1 - X_2) \subseteq \mathcal{P}_2$, then $\mathfrak{n}_V = 1 < \infty = \varkappa_V$).

Lemma 25. *Let V be a proper ideal in $\mathcal{P}_N$ and Q be a column polynomial. Then*

(i) $\deg_V Q = \deg Q$ *provided* $\deg Q < \mathfrak{n}_V$.

If moreover V is a $*$-ideal and Q is real, then

(ii) *there exists a real column polynomial $\widetilde{Q}$ such that $Q \overset{V}{=} \widetilde{Q}$ and* $\deg_V Q = \deg \widetilde{Q}$,

(iii) *for every integer $j \geqslant \mathfrak{n}_V$, there exists a real column polynomial R such that $R \overset{V}{=} Q$ and all the entries of $R - Q$ are polynomials of degree j.*

Proof. (i) Take a column polynomial P such that $Q \overset{V}{=} P$ and $\deg_V Q = \deg P$. Then $Q - P \subseteq V$ and $\deg(Q - P) < \mathfrak{n}_V$ (because $\deg_V Q \leqslant \deg Q$). Hence $Q = P$.

(ii) If P is any column polynomial such that and $Q \overset{V}{=} P$ and $\deg_V Q = \deg P$, then the real column polynomial $\widetilde{Q} \overset{\text{df}}{=} \frac{1}{2}(P + P^{*\mathsf{T}})$ has the desired properties.

(iii) We have only to consider the case $V \neq \{0\}$. Suppose first that $\ell(Q) = 1$. Taking $p \in V \setminus \{0\}$ such that $\deg p = \mathfrak{n}_V$, we see that either $\mathfrak{Re}\, p$ or $\mathfrak{Im}\, p$ is a polynomial of degree $\mathfrak{n}_V$, which belongs to V. This means that there exists a real polynomial $q \in V$ of degree $\mathfrak{n}_V$. As a consequence, the polynomial $R \overset{\text{df}}{=} Q + X_1^{j - \mathfrak{n}_V} q$ is the desired one. If $\ell(Q) > 1$, then we proceed entrywise. $\square$

Below we show that the degree requirement is stronger than the rank condition.

Proposition 26. *Let V be a proper $*$-ideal in $\mathcal{P}_N$ and $\{Q_k\}_{k=0}^{\varkappa_V}$ be a sequence of column polynomials satisfying condition* (B-i) *of Theorem* 18 *and* $\deg_V Q_0 = 0$. *Then the following conditions are equivalent*:

(i) $\deg_V Q_k \leqslant k$ *for all* $k \in \overline{0, \varkappa_V}$ *and* (B-ii) *holds*;
(ii) $[A_{k,1}^*, \ldots, A_{k,N}^*]^*$ *is injective for all* $k \in \overline{0, \varkappa_V}$ *and* (B-ii) *holds*;
(iii) $[A_{k,1}^*, \ldots, A_{k,N}^*]^*$ *is of maximal rank and* $\ell(Q_k) = d_V(k)$ *for all* $k \in \overline{0, \varkappa_V}$.

If moreover $V = \{0\}$, *then any of conditions* (ii) *and* (iii) *is equivalent to the conjunction of* (B-ii) *and* (B-iii).

Proof. (i)⇒(iii) Replacing, if necessary, $\{Q_k\}_{k=0}^{\varkappa_V}$ by a new sequence $\{P_k\}_{k=0}^{\varkappa_V}$ such that $Q_k \overset{V}{=} P_k$ and $\deg_V Q_k = \deg P_k$ for all $k \in \overline{0, \varkappa_V}$, we can assume without loss of generality that $\deg Q_k \leqslant k$ for all $k \in \overline{0, \varkappa_V}$ and $Q_0 \neq 0$. Applying Step 2 of the proof of Theorem 18, we get (iii) (recall that in Step 2 the Q_k's need not be real).

(iii)⇒(ii) This is a direct consequence of (B-i) and (8).

(ii)⇒(i) Fix an integer $0 \leqslant k < \varkappa_V$. Multiplying both sides of (25) by the left inverse of $[A_{k,1}^*, \dots, A_{k,N}^*]^*$, we find scalar matrices $\{D_n\}_{n=1}^N$ and $\{E_l\}_{l=0}^k$ such that (37) holds. Proceeding by induction on k and using the assumptions that V is an ideal and $\deg_V Q_0 = 0$, we get $\deg_V Q_k \leqslant k$ for every $k \in \overline{0, \varkappa_V}$. □

Remark 27. Let V, L and $\{Q_k\}_{k=0}^n$ satisfy the assumptions of Theorem 18 and let $\deg_V Q_0 = 0$. We say that $\{Q_k\}_{k=0}^n$ satisfies (A*) if there exists a sequence $\{\widetilde{Q}_k\}_{k=0}^n$ of real column polynomials satisfying (A) and $Q_j \overset{V}{=} \widetilde{Q}_j$ for all $j \in \overline{0, n}$. By (B*) we mean the conjunction of conditions (B-i), (B-ii), (B-iii*) and (B-iv) with $n = \varkappa_V$, where (B-iii*) is defined by:

(B-iii*) the matrix $[A_{k,1}^*, \dots, A_{k,N}^*]^*$ is injective for every $k \in \overline{0, n}$.

Conditions (A*) and (B*) are weaker than (A) and (B), respectively (cf. Proposition 26). Consider a sequence $\{\widetilde{Q}_k\}_{k=0}^n$ of real column polynomials such that $Q_j \overset{V}{=} \widetilde{Q}_j$ for all $j \in \overline{0, n}$. It is clear that if $\{Q_k\}_{k=0}^n$ satisfies condition (B-i) (resp. (B-ii)), then so does $\{\widetilde{Q}_k\}_{k=0}^n$ with the same system of matrices $A_{k,j}, B_{k,j}, C_{k,j}$. The same is true for any of conditions (B-iii*) and (B-iv) provided (B-i) holds. Likewise, if $\{Q_k\}_{k=0}^n$ is a V-basis of $\mathcal{P}_N$ (resp. it is quasi-orthogonal with respect to a linear functional L vanishing on V), then so is $\{\widetilde{Q}_k\}_{k=0}^n$. On the other hand, if $\{Q_k\}_{k=0}^n$ is a rigid V-basis of $\mathcal{P}_N$ (resp. it satisfies (B-iii)), then $\{\widetilde{Q}_k\}_{k=0}^n$ does not have to share this property. This explains why replacing (B-iii) by (B-iii*) makes the implication (B)⇒(A) false. However, the following is true.

If $\deg_V Q_0 = 0$, then $\{Q_k\}_{k=0}^n$ satisfies (B*) if and only if $\{Q_k\}_{k=0}^n$ satisfies (A*) with some L which is unique up to a multiplicative constant. (40)

Indeed, if (B*) holds, then by part (ii) of Lemma 25 there exists a sequence $\{\widetilde{Q}_k\}_{k=0}^n$ of real column polynomials such that $Q_j \overset{V}{=} \widetilde{Q}_j$ and $\deg_V Q_j = \deg \widetilde{Q}_j$ for all $j \in \overline{0, n}$. Hence, by Proposition 26, $\{\widetilde{Q}_k\}_{k=0}^n$ satisfies (B) and consequently, by Theorem 18, it satisfies (A). The reverse implication can be proved in a similar manner with the help of Theorem 18.

In view of (40), condition (B*) should have been formulated rather for sequences of equivalence classes (modulo V) of polynomials than for sequences of polynomials themselves.

Concluding this remark, we notice that if $V \neq \{0\}$, $\{Q_k\}_{k=0}^n$ is a sequence of real column polynomials satisfying (B) (with $Q_0 \neq 0$) and $\{m_k\}_{k=0}^n \subseteq \mathbb{N}$ is such that $m_k \geqslant \max\{k, \mathfrak{n}_V\}$ for all $k \in \overline{0, n}$, then there exists a V-basis $\{\widetilde{Q}_k\}_{k=0}^n$ of $\mathcal{P}_N$ composed of real column polynomials satisfying (B*) and having the property that for every $k \in \overline{0, n}$, each entry of $\widetilde{Q}_k$ is of degree m_k (this means that (B*)⇏(A)). Indeed, we may define $\{\widetilde{Q}_k\}_{k=0}^n$ via

$$\widetilde{Q}_k = \begin{cases} R_k & \text{if } m_k > k, \\ Q_k & \text{if } m_k = k, \end{cases}$$

where R_k is a real column polynomial such that $R_k \stackrel{V}{=} Q_k$ and all the entries of $R_k - Q_k$ are of degree m_k (cf. part (iii) of Lemma 25). By part (a) of Theorem 18, the sequence $\{\widetilde{Q}_k\}_{k=0}^n$ has the desired properties.

Applying our Theorem 18 and Proposition 26, we show that Theorem 2 of [31] can be simplified by replacing one of its rank assumptions by the requirement on degrees of polynomials in question. By a *rigid basis* of $\mathcal{R}_N$ we mean a basis $\{P_k\}_{k=0}^\infty$ of $\mathcal{R}_N$, which is simultaneously a rigid basis of $\mathcal{P}_N$; equivalently: $\{P_k\}_{k=0}^\infty$ is a rigid basis of $\mathcal{P}_N$ composed of real column polynomials. Given a linear functional $L : \mathcal{R}_N \to \mathbb{R}$, we define the mapping $\mathcal{R}_N \times \mathcal{R}_N \ni (p,q) \longmapsto \langle p,q\rangle_L \stackrel{\mathrm{df}}{=} L(pq) \in \mathbb{R}$. Following [31], we say that $\langle \cdot, - \rangle_L$ is a *quasi-inner product* on $\mathcal{R}_N$ if there exists a rigid basis $\{P_k\}_{k=0}^\infty$ of $\mathcal{R}_N$ such that $L(P_i P_j^\mathsf{T}) = 0$ for all $i \neq j$ and $L(P_k P_k^\mathsf{T})$ is a non-singular diagonal matrix for every $k \in \mathbb{N}$.

Corollary 28. *Let $\{P_k\}_{k=0}^\infty$ be a sequence of real column polynomials such that $P_0 \neq 0$. Then the following conditions are equivalent*:

(i) *$\{P_k\}_{k=0}^\infty$ is a rigid basis of $\mathcal{R}_N$ for which there exists a linear functional $L : \mathcal{R}_N \to \mathbb{R}$ such that $\langle \cdot, - \rangle_L$ is a quasi-inner product on $\mathcal{R}_N$ and $L(P_i P_j^\mathsf{T}) = 0$ for all $i \neq j$;*

(ii) *$\{P_k\}_{k=0}^\infty$ is a rigid basis of $\mathcal{R}_N$ for which there exists a linear functional $L : \mathcal{R}_N \to \mathbb{R}$ such that $L(P_i P_j^\mathsf{T}) = 0$ for all $i \neq j$ and $L(P_k P_k^\mathsf{T})$ is non-singular for every $k \geqslant 0$;*

(iii) *for every $k \in \mathbb{N}$, there exists a system $A_{k,1}, \ldots, A_{k,N}, B_{k,1}, \ldots, B_{k,N}, C_{k,1}, \ldots, C_{k,N}$ of scalar real matrices such that*

  (iii-a) *$X_j P_k = A_{k,j} P_{k+1} + B_{k,j} P_k + C_{k,j} P_{k-1}$ for all $j = 1, \ldots, N$, where $C_{0,j} \stackrel{\mathrm{df}}{=} 1$ and $P_{-1} \stackrel{\mathrm{df}}{=} 0$,*

  (iii-b) *the length of P_k is less than or equal to $\binom{k+N-1}{k}$,*

  (iii-c) deg $P_k \leqslant k$,

  (iii-d) *the matrix $[C_{k,1}, \ldots, C_{k,N}]$ is of maximal rank.*

If (i) *holds, then for every $p \in \mathcal{R}_N$, $p = 0$ if and only if $L(pq) = 0$ for all $q \in \mathcal{R}_N$. If* (iii) *holds, then $[A_{k,1}^\mathsf{T}, \ldots, A_{k,N}^\mathsf{T}]^\mathsf{T}$ is of maximal rank for all $k \in \mathbb{N}$.*

Proof. Apply Theorem 18, Corollary 22 and part (a) of Remark 19 to the Hermitian (complex) linear functional $\mathcal{P}_N \ni p \longmapsto L_\mathbb{C}(p) \stackrel{\mathrm{df}}{=} L(\mathfrak{Re}\,p) + \mathrm{i}L(\mathfrak{Im}\,p) \in \mathbb{C}$ and to the $*$-ideal $V = \{0\}$. Notice also that

$$\mathcal{V}_{L_\mathbb{C}} = \{0\} \quad \text{if and only if} \quad \{p \in \mathcal{R}_N : L(pq) = 0 \quad \forall\, q \in \mathcal{R}_N\} = \{0\}. \qquad \square$$

7. Quasi-orthogonality: the complex case

Our aim in this section is to prove a version of Theorem 18 for polynomials which are not assumed to be real. We begin by formulating appropriate criteria for a linear functional on $\mathcal{P}_N$ to be Hermitian.

Lemma 29. *Let V be a proper $*$-ideal, $L : \mathcal{P}_N \to \mathbb{C}$ be a linear functional and $\{Q_k\}_{k=0}^n$ $(0 \leqslant n \leqslant \infty)$ be a V-basis of $\mathcal{P}_N$ such that $Q_0 \in \mathbb{C}$, $L(Q_j) = 0$ for every $j \in \overline{1, n}$, $L(X^0) \in \mathbb{R} \setminus \{0\}$ and $V \subseteq \ker L$.*

Consider the following two conditions:

(C-i) *for every $k \in \overline{0, n}$, there exist a non-singular scalar matrix W_k and a real column polynomial S_k such that $W_k Q_k \overset{V}{=} S_k$,*

(C-ii) *for every $k \in \overline{1, n}$, $\Pi_V(Q_k^*) \subseteq \operatorname{lin} \Pi_V\big(\bigcup_{i=1}^n Q_i\big)$.*

Then (C-i) *implies* (C-ii), *while* (C-ii) *is equivalent to L being Hermitian. Moreover*, (C-ii) *implies* (C-i) *provided condition* (A) *of Theorem* 18 *is satisfied.*

Proof. Assume that (C-i) holds. Fix $k \in \overline{0, n}$. Since V is a $*$-ideal and S_k is a real column polynomial, we get

$$Q_k^* W_k^* = (W_k Q_k)^* \overset{V}{=} S_k^{\mathsf{T}} \overset{V}{=} (W_k Q_k)^{\mathsf{T}} = Q_k^{\mathsf{T}} W_k^{\mathsf{T}}$$

and consequently $Q_k^* \overset{V}{=} Q_k^{\mathsf{T}} W_k^{\mathsf{T}} (W_k^*)^{-1}$. This implies (C-ii).

Suppose (C-ii) is satisfied. Fix $k \in \overline{1, n}$. By Lemma 3, there exists a finite sequence $\{D_i\}_{i=1}^n$ of scalar matrices such that $Q_k^{*\mathsf{T}} \overset{V}{=} \sum_{i=1}^n D_i Q_i$. This leads to

$$L(Q_k^*)^{\mathsf{T}} = L(Q_k^{*\mathsf{T}}) = \sum_{i=1}^n D_i L(Q_i) = 0. \tag{41}$$

Take $p \in \mathcal{P}_N$. Then $p = p_1 + p_2$, where $p_1 \in V$ and $p_2 \in \operatorname{lin} \bigcup_{i=0}^n Q_i$. By $V = V^*$, $L(p_1^*) = 0 = L(p_1)$. Since $\bigcup_{i=0}^n Q_i$ is a basis of $\operatorname{lin} \bigcup_{i=0}^n Q_i$, there exists a finite sequence $\{E_i\}_{i=0}^n$ of scalar rows such that $p_2 = \sum_{i=0}^n E_i Q_i$. By (41), we have

$$L(p_2^*) = L\left(\sum_{i=0}^n Q_i^* E_i^*\right) = L(Q_0^*) E_0^* = Q_0^* L(X^0) E_0^* = \overline{L(p_2)}.$$

This implies that the functional L is Hermitian.

Assume now that L is Hermitian. Fix $k \in \overline{1, n}$. Then there exists a finite sequence $\{D_i\}_{i=0}^n$ of scalar matrices such that $Q_k^{*\mathsf{T}} \overset{V}{=} \sum_{i=0}^n D_i Q_i$. Thus we have

$$0 = L(Q_k)^{*\mathsf{T}} = L(Q_k^{*\mathsf{T}}) = D_0 Q_0 L(X^0).$$

Since $Q_0 L(X^0) \neq 0$, we conclude that $D_0 = 0$. This gives us (C-ii).

To prove the last assertion, we show that if L is a Hermitian linear functional satisfying (A), then (C-i) holds with $W_k \overset{\text{df}}{=} G_k^{-1}$, where G_k are non-singular scalar matrices appearing in part (b) of Proposition 9. Since, by Proposition 9, $\{G_k^{-1} Q_k\}_{k=0}^{\varkappa_V}$ is a rigid V-basis of $\mathcal{P}_N$, we can assume without loss of generality (replacing $\{Q_k\}_{k=0}^{\varkappa_V}$ by $\{G_k^{-1} Q_k\}_{k=0}^{\varkappa_V}$) that each G_k is the identity matrix. Thus for each $k \in \overline{0, \varkappa_V}$, there exists a unique system $\{D_j^{(k)}\}_{j=0}^{k-1}$

of scalar matrices such that

$$Q_k \overset{V}{=} \Sigma_k^V - \sum_{j=0}^{k-1} D_j^{(k)} Q_j. \tag{42}$$

Using induction, we show that for every $k \in \overline{0, \varkappa_V}$, there exists a real column polynomial S_k such that $Q_k \overset{V}{=} S_k$. The case $k = 0$ is trivial ($S_0 \overset{\text{df}}{=} X^0$). Assume we have constructed $S_0, \ldots, S_{k-1}$ with the desired properties ($k \in \overline{1, \varkappa_V}$). Set $S_k = \Sigma_k^V - \sum_{j=0}^{k-1} D_j^{(k)} S_j$. It is evident by (42) that $Q_k \overset{V}{=} S_k$. It remains to prove that all the scalar matrices $\{D_j^{(k)}\}_{j=0}^{k-1}$ are real. Fix $i \in \overline{0, k-1}$. By Theorem 18, the matrix $L(Q_i Q_i^*)$ is non-singular. Multiplying both sides of (42) by Q_i^*, we get $L(\Sigma_k^V Q_i^*) = D_i^{(k)} L(Q_i Q_i^*)$. This and the equalities $Q_i \overset{V}{=} S_i$ and $Q_i^* \overset{V}{=} S_i^\mathsf{T}$ give us $D_i^{(k)} = L(\Sigma_k^V Q_i^*) L(Q_i Q_i^*)^{-1} = L(\Sigma_k^V S_i^\mathsf{T}) L(S_i S_i^\mathsf{T})^{-1}$. Since L is Hermitian, we conclude that S_k is a real column polynomial. □

We are now in a position to prove a "complex" version of Theorem 18.

Theorem 30. *Let V be a proper $*$-ideal in $\mathcal{P}_N$, $L : \mathcal{P}_N \to \mathbb{C}$ be a linear functional and $\{Q_k\}_{k=0}^n$ $(0 \leqslant n \leqslant \infty)$ be a sequence of column polynomials such that $Q_0 \neq 0$. Let* (A), (B) *be as in Theorem* 18, *and* (C-i), (C-ii) *be as in Lemma* 29 *with $n = \varkappa_V$.*

Then

(a) (B-i), (B-ii) *and* (B-iii) *imply that $[A_{k,1}^*, \ldots, A_{k,N}^*]^*$ is injective for every $k \in \overline{0, \varkappa_V}$ and $\{Q_k\}_{k=0}^n$ is a rigid V-basis of $\mathcal{P}_N$; if, in addition to* (B-i), (B-ii) *and* (B-iii), *any of the two conditions* (C-i) *and* (C-ii) *holds, then L defined by* (32) *is Hermitian and $L(Q_i Q_j^*) = 0$ for all $i \neq j$,*

(b) (B) *together with any of conditions* (C-i) *and* (C-ii) *imply* (A) *and the non-singularity of $L(Q_k Q_k^*)$ for all $k \in \overline{0, \varkappa_V}$, where L is defined by* (32),

(c) (A) *implies* (B), (C-i) *and* (C-ii), *provided L is Hermitian*; *if a linear functional L' : $\mathcal{P}_N \to \mathbb{C}$ satisfies* (A), *then $L'(X^0) \neq 0$ and the functional $\frac{1}{L'(X^0)} L'$ fulfills* (32) (*L' is not assumed to be Hermitian*).

Proof. The proof of parts (a) and (b) of the conclusion is essentially the same as that in Theorem 18. We only have to modify the proof of Step 3 in order to show that L defined by (32) is Hermitian without referring to $\{Q_k\}_{k=0}^{\varkappa_V}$ being real column polynomials. However, Lemma 29 guarantees that any of the two conditions (C-i) and (C-ii) implies L being Hermitian.

All the remaining statements of the conclusion may be justified in much the same way as it has been done in the proof of Theorem 18; however, we need to apply Lemma 29 to show that (A) implies (C-i) and (C-ii) provided L is Hermitian (notice that $V = \mathcal{V}_L$ being proper excludes $L(X^0) = 0$). □

Remark 31. Let (A), (B) be as in Theorem 18 and (C-i), (C-ii) be as in Lemma 29 with $n = \varkappa_V$. A careful inspection of the proof of Lemma 29 shows that if the functional

L is Hermitian, then (A) implies (C-ii) with $\operatorname{lin}\Pi_V(Q_k^*) = \operatorname{lin}\Pi_V(Q_k)$, and (C-i) with $W_k = G_k^{-1}$ and $\deg S_k = k$, where G_k is a non-singular scalar matrix appearing in part (b) of Proposition 9. Equivalent forms of (A) are contained in Proposition 21, while some facts related to (B) can be found in Proposition 26.

The reader may have noticed that Proposition 13 is formulated for rigid bases of $\mathcal{P}_N$, while Theorems 18 and 30 concern rigid V-bases of $\mathcal{P}_N$. However, by Proposition 10, there is no loss of generality in assuming that the sequence $\{Q_k\}_{k=0}^n$ appearing in both these theorems is selected from a rigid basis $\{P_k\}_{k=0}^\infty$ of $\mathcal{P}_N$ so that $\bigcup_{k=0}^\infty P_k \setminus \bigcup_{k=0}^n Q_k \subseteq V$.

8. Orthogonality

In this section we restrict our attention to orthogonality of polynomials of several variables with respect to a positive definite linear functional. We will state and prove refined versions of the main results of Sections 5 and 7.

Let $L : \mathcal{P}_N \to \mathbb{C}$ be a *positive definite* linear functional, i.e. $L(pp^*) \geqslant 0$ for all $p \in \mathcal{P}_N$. It is well known that such L has to be Hermitian (cf. [9, Lemma V.37.6]). Applying the Cauchy–Schwarz inequality to the semi-inner product $(p, q) \longmapsto L(pq^*)$ on $\mathcal{P}_N$, we get

$$\mathcal{V}_L = \{p \in \mathcal{P}_N : L(pp^*) = 0\}. \tag{43}$$

Suppose now that $L : \mathcal{P}_N \to \mathbb{C}$ is a linear functional. A sequence $\{Q_k\}_{k=0}^n$ $(0 \leqslant n \leqslant \infty)$ of column polynomials is said to be *L-orthonormal* if $L(Q_i Q_j^*) = 0$ for all $i \neq j$, and $L(Q_k Q_k^*)$ is the identity matrix for every $k \in \overline{0, n}$. Notice that each L-orthonormal sequence $\{Q_k\}_{k=0}^n$ is linearly V-independent for any ideal $V \subseteq \mathcal{V}_L$. Indeed, if $\sum_{i=0}^n D_i Q_i \overset{V}{=} 0$, where $\{D_i\}_{i=0}^n$ is a finite sequence of scalar rows, then $D_j = L\big(\big(\sum_{i=0}^n D_i Q_i\big) Q_j^*\big) = 0$ for all $j \in \overline{0, n}$. This and Lemma 3 give us the desired V-independence.

The following proposition provides necessary and sufficient conditions for a linear functional to be positive definite.

Proposition 32. *If $L : \mathcal{P}_N \to \mathbb{C}$ is a non-zero linear functional, then the following conditions are equivalent*

(i) *L is positive definite,*
(ii) *$\mathcal{V}_L$ is a $*$-ideal and there is a rigid $\mathcal{V}_L$-basis of $\mathcal{P}_N$, which is L-orthonormal,*
(iii) *$\mathcal{V}_L$ is a $*$-ideal and there is a $\mathcal{V}_L$-basis of $\mathcal{P}_N$, which is L-orthonormal,*
(iv) *there is a basis B of $\mathcal{P}_N$ such that $L(pp^*) \in \{0, 1\}$ and $L(qr^*) = 0$ for all $p, q, r \in B$ such that $q \neq r$.*

Proof. Set $V = \mathcal{V}_L$. Since L is non-zero, the ideal V is proper.

(i)⇒(ii) The functional L being positive definite is Hermitian. Hence the set V is a $*$-ideal. Since $\{\Sigma_k^V\}_{k=0}^{\varkappa_V}$ is a V-basis of $\mathcal{P}_N$, the set $\Lambda_N^V = \bigcup_{k=0}^{\varkappa_V} \Sigma_k^V$ is a basis of $F \overset{\text{df}}{=} \operatorname{lin}\Lambda_N^V$ and $\mathcal{P}_N$ is the direct sum of V and F. This and (43) imply that the mapping $F \times F \ni (p, q) \longmapsto \langle p, q\rangle_L \overset{\text{df}}{=} L(pq^*) \in \mathbb{C}$ is an inner product on F. Arrange members of Λ_N^V in a sequence

$\{X^{\alpha_k}\}_{k=1}^s$ so that $\alpha_1 = 0$ and[6] $\alpha_k \lessdot \alpha_{k+1}$ for every $k \in \overline{1,s}$ such that $k+1 \leqslant s$, where $s \stackrel{\mathrm{df}}{=} \sum_{j=0}^{\varkappa_V} d_V(j)$ (this means that $\Sigma_k^V = \{X^{\alpha_i} : |\alpha_i| = k\}$ for $k \in \overline{0, \varkappa_V}$). Applying the Gram–Schmidt orthonormalization procedure to $\{X^{\alpha_k}\}_{k=1}^s$ with respect to the inner product $\langle \cdot, - \rangle_L$, we find a basis $\{q_j\}_{j=1}^s$ of F such that

$$\operatorname{lin}\{X^{\alpha_i} : i \in \overline{1,j}\} = \operatorname{lin}\{q_i : i \in \overline{1,j}\}, \quad j \in \overline{1,s}, \tag{44}$$

$$L(q_i q_j^*) = \delta_{i,j}, \quad i,j \in \overline{1,s}. \tag{45}$$

Set $Q_k = \{q_i : |\alpha_i| = k\}$ for $k \in \overline{0, \varkappa_V}$. By (44), $\deg q_j \leqslant |\alpha_j|$ for $j \in \overline{1,s}$, and consequently $\deg Q_k \leqslant k$ for $k \in \overline{0, \varkappa_V}$. According to (7), $\ell(Q_k) = d_V(k)$ for $k \in \overline{0, \varkappa_V}$. Since a subset C of F is linearly independent if and only if C is linearly V-independent, we conclude that the sequence $\{Q_k\}_{k=0}^{\varkappa_V}$ is linearly V-independent. Hence, by (45) and part (i) of Proposition 6, $\{Q_k\}_{k=0}^{\varkappa_V}$ is a rigid V-basis of $\mathcal{P}_N$, which is L-orthonormal.

(ii)⇒(iii) Trivial.

(iii)⇒(iv) Assume that $\{Q_k\}_{k=0}^n$ $(0 \leqslant n \leqslant \infty)$ is a V-basis of $\mathcal{P}_N$, which is L-orthonormal. Then $C \stackrel{\mathrm{df}}{=} \bigcup_{k=0}^n Q_k$ is a basis of $F \stackrel{\mathrm{df}}{=} \operatorname{lin} C$, and $\mathcal{P}_N$ is the direct sum of V and F. Let D be a basis of V. Then $B \stackrel{\mathrm{df}}{=} C \cup D$ is a basis of $\mathcal{P}_N$. One can deduce from $V = V^*$ and the L-orthonormality of $\{Q_k\}_{k=0}^n$ that B has all the desired properties.

(iv)⇒(i) If $r = \sum_{p \in B} \alpha_p p$, where $\{\alpha_p\}_{p \in B} \subseteq \mathbb{C}$ is a finite system, then

$$L(rr^*) = L\left(\left(\sum_{p \in B} \alpha_p p\right)\left(\sum_{q \in B} \bar{\alpha}_q q^*\right)\right) = \sum_{p \in B} |\alpha_p|^2 L(pp^*) \geqslant 0,$$

which proves the positive definiteness of L. □

Remark 33. Let L be a non-zero positive definite linear functional on $\mathcal{P}_N$. A thorough inspection of the proof reveals that all the bases appearing in conditions (ii), (iii) and (iv) of Proposition 32 can be chosen so as to be composed of real polynomials. Case (ii) may be handled with the help of the Gram–Schmidt orthonormalization procedure given by the following explicit formulas: $q_1 = \frac{1}{\sqrt{G_1}} X^{\alpha_1}$ and

$$q_n = \frac{1}{\sqrt{G_n G_{n-1}}} \det \begin{bmatrix} L(X^{\alpha_1} X^{\alpha_1}) & \cdots & L(X^{\alpha_1} X^{\alpha_{n-1}}) & X^{\alpha_1} \\ L(X^{\alpha_2} X^{\alpha_1}) & \cdots & L(X^{\alpha_2} X^{\alpha_{n-1}}) & X^{\alpha_2} \\ \vdots & & \vdots & \vdots \\ L(X^{\alpha_n} X^{\alpha_1}) & \cdots & L(X^{\alpha_n} X^{\alpha_{n-1}}) & X^{\alpha_n} \end{bmatrix} \quad \text{for } n \in \overline{2,s},$$

where $G_n \stackrel{\mathrm{df}}{=} \det[L(X^{\alpha_i} X^{\alpha_j})]_{i,j=1}^n$ for $n \in \overline{1,s}$. Since monomials $\{X^{\alpha_k}\}_{k=1}^s$ are linearly independent, their Gramians $\{G_k\}_{k=1}^s$ are positive. As L is Hermitian, all the polynomials $\{q_k\}_{k=1}^s$ are real. Case (iii) is covered by (ii), while (iv) requires showing that there exists a (linear) basis of $\mathcal{V}_L$ composed of real polynomials (this property is shared by all $*$-ideals in $\mathcal{P}_N$). Indeed, if D is an arbitrary (linear) basis of $\mathcal{V}_L$, then $\mathcal{V}_L = \operatorname{lin}\big(\{\mathfrak{Re}\, p : p \in$

[6] See (1) for the definition of $\lessdot$.

$D\} \cup \{\mathfrak{Im}\, p : p \in D\})$ and consequently the desired basis can be selected from the set $\{\mathfrak{Re}\, p : p \in D\} \cup \{\mathfrak{Im}\, p : p \in D\}$. Notice finally that in general neither $\{\mathfrak{Re}\, p : p \in D\}$ nor $\{\mathfrak{Im}\, p : p \in D\}$ has to be a basis of $\mathcal{V}_L$, e.g. $L(p) = p(0)$ for $p \in \mathcal{P}_1$ and $D = \{\mathrm{i}^k(X^k + \mathrm{i}X)\}_{k=1}^{\infty}$.

Example 34. If we drop the assumption that $\mathcal{V}_L$ is a $*$-ideal in any of the conditions (ii) and (iii) of Proposition 32, then the functional L may not be positive definite. Let $\alpha_1, \alpha_2, z_1, z_2$ be complex numbers such that $\alpha_1\alpha_2 \neq 0$, $\alpha_1+\alpha_2 = 1$, $\beta \stackrel{\text{df}}{=} \alpha_1 z_1 + \alpha_2 z_2 \in \mathbb{R}$, $\gamma \stackrel{\text{df}}{=} \alpha_1 z_1^2 + \alpha_2 z_2^2 - \beta^2 > 0$, $z_1 \neq z_2$ and $\{z_1, z_2\} \neq \{\bar{z}_1, \bar{z}_2\}$ (e.g. $\alpha_1 = 2$, $\alpha_2 = -1$, $z_1 = 1+\mathrm{i}$ and $z_2 = 1+2\mathrm{i}$). Define the linear functional L on $\mathcal{P}_1$ by $L(p) = \alpha_1 p(z_1) + \alpha_2 p(z_2)$ for $p \in \mathcal{P}_1$. One can check that $\{X^0, X\}$ is a $\mathcal{V}_L$-basis of $\mathcal{P}_1$ and $\mathcal{V}_L = \{p \in \mathcal{P}_1 : p(z_1) = p(z_2) = 0\}$. Hence $\mathcal{V}_L$ is not a $*$-ideal and, in consequence, L is not positive definite. Notice also that $\dim \mathcal{P}_1/\mathcal{V}_L = 2$, $d_V(0) = d_V(1) = 1$ and $\varkappa_{\mathcal{V}_L} = 1$. Set $Q_0 = X^0$ and $Q_1 = \frac{1}{\sqrt{\gamma}}(X - \beta)$. A straightforward computation shows that $L(Q_0 Q_0^*) = 1$, $L(Q_0 Q_1^*) = L(Q_1 Q_0^*) = 0$ and $L(Q_1 Q_1^*) = 1$. This implies that $\{Q_k\}_{k=0}^{\varkappa_{\mathcal{V}_L}}$ is a rigid $\mathcal{V}_L$-basis of $\mathcal{P}_1$, which is an L-orthonormal set of real polynomials. We believe that this idea should work for $\varkappa_{\mathcal{V}_L} > 1$ as well.

Remark 35. By Propositions 16 and 32 we infer that if $L : \mathcal{P}_N \to \mathbb{C}$ is a non-zero positive definite linear functional, then the matrix $L(\Xi_k^{\{0\}}(\Xi_k^{\{0\}})^*)$ is of rank $d_V(0) + \cdots + d_V(k)$ for $k \in \mathbb{N}$, where $V = \mathcal{V}_L$ and $\Xi_k^{\{0\}}$ is given by (29). This formula may be useful to calculate the exact values of $d_V(k)$ at least in the case of set ideals (cf. Section 9).

The result which follows solves the question of orthonormality with respect to a positive definite linear functional. Notice the absence of the rank condition in part (B), in contrast to Theorem 18.

Theorem 36. *Let V be a proper $*$-ideal in $\mathcal{P}_N$, $L : \mathcal{P}_N \to \mathbb{C}$ be a linear functional and $\{Q_k\}_{k=0}^n$ $(0 \leqslant n \leqslant \infty)$ be a sequence of real column polynomials such that $Q_0 = 1$. Consider the following two conditions*:[7]

(A) *$\{Q_k\}_{k=0}^n$ is a rigid V-basis of $\mathcal{P}_N$, which is L-orthonormal, and $V \subseteq \ker L$;*

(B) *$n = \varkappa_V$ and there exists a system $\{[A_{k,j}, B_{k,j}]\}_{k=0}^{\varkappa_V}{}_{j=1}^{N}$ of scalar matrices such that*

- (B-i) $X_j Q_k \stackrel{V}{=} A_{k,j} Q_{k+1} + B_{k,j} Q_k + A_{k-1,j}^* Q_{k-1}$ *for all $j \in \overline{1, N}$ and $k \in \overline{0, \varkappa_V}$, where $A_{-1,j} \stackrel{\text{df}}{=} 1$ and $Q_{-1} \stackrel{\text{df}}{=} 0$; if $\varkappa_V < \infty$, then $A_{\varkappa_V, j} \stackrel{\text{df}}{=} [1, \ldots, 1]^*$ with $\ell(A_{\varkappa_V, j}) = \ell(Q_{\varkappa_V})$ and $Q_{\varkappa_V+1} \stackrel{\text{df}}{=} 0$,*
- (B-ii) *the length of Q_k is less than or equal to $d_V(k)$ for every $k \in \overline{0, \varkappa_V}$,*
- (B-iii) $\deg Q_k \leqslant k$ *for every $k \in \overline{0, \varkappa_V}$.*

Then (B) *implies* (A), *the injectivity of $[A_{k,1}^*, \ldots, A_{k,N}^*]^*$ for every $k \in \overline{0, \varkappa_V}$ and the positive definiteness of L, where L is defined by* (32). *Conversely,* (A)

[7] See Proposition 26 for related facts concerning (B-iii).

implies (B) *and* $V = \mathcal{V}_L$. *If a linear functional* $L' : \mathcal{P}_N \to \mathbb{C}$ *satisfies* (A), *then* L' *fulfills* (32).

Proof. (B)⇒(A) It follows from Theorem 18 that the matrix $[A_{k,1}^*, \ldots, A_{k,N}^*]^*$ is injective. This means that part (B-iv) of Theorem 18 is satisfied with $C_{k,j} = A_{k-1,j}^*$. Applying once more Theorem 18, we conclude that $V = \mathcal{V}_L$, $\{Q_k\}_{k=0}^{\varkappa_V}$ is a rigid V-basis of $\mathcal{P}_N$ and $L(Q_k Q_j^*) = 0$ for all $k \neq j$, where L is the Hermitian linear functional defined by (32). Using induction we show that $L(Q_k Q_k^*)$ is the identity matrix. If $k = 0$, then $L(Q_0 Q_0^*) = 1$ because of (32). Suppose that the induction hypothesis holds for a fixed k. Then by the injectivity of $[A_{k,1}^*, \ldots, A_{k,N}^*]^*$ and (28), $L(Q_{k+1} Q_{k+1}^*)$ is the identity matrix as well, which completes the induction argument. Positive definiteness of L is now guaranteed by Proposition 32.

(A)⇒(B) By Proposition 21 and Theorem 18, it remains to show that $C_{k,j} = A_{k-1,j}^*$ for all $k \in \overline{0, \varkappa_V}$ and $j \in \overline{1, N}$. However this follows from (28) and the assumption that each $L(Q_k Q_k^*)$ is the identity matrix.

Since $L'(X^0) = L'(Q_0 Q_0^*) = 1$, the last assertion is forced by Theorem 18. □

Exploiting Theorem 36, one can formulate a simplified version of Theorem 2.2 of [32] (compare with Corollary 28; see also Proposition 26). Once more the rank condition can be replaced by the assumption on degrees of polynomials involved.

Analogous to Theorem 30, we can state a "complex" version of Theorem 36.

Theorem 37. *Let* V *be a proper* $*$*-ideal in* $\mathcal{P}_N$, $L : \mathcal{P}_N \to \mathbb{C}$ *be a linear functional and* $\{Q_k\}_{k=0}^{n}$ $(0 \leqslant n \leqslant \infty)$ *be a sequence of column polynomials such that* $Q_0 = 1$. *Let* (A), (B) *be as in Theorem* 36, *and* (C-i), (C-ii) *be as in Lemma* 29 *with* $n = \varkappa_V$.

Then the whole conclusion of Theorem 36 *remains true provided* (B) *is strengthened by either of the two conditions* (C-i) *and* (C-ii).

Proof. According to Propositions 21 and 32, (A) implies that $V = \mathcal{V}_L$ and L is Hermitian. This enables us to repeat arguments used in the proof of Theorem 36 replacing Theorem 18 by Theorem 30. □

9. Algebraic sets as supports of orthogonalizing measures

Denote by $\mathfrak{M}_N$ the set of all positive Borel measures μ on $\mathbb{R}^N$ with all finite moments, i.e. $\int_{\mathbb{R}^N} |x^\alpha| \, d\mu(x) < \infty$ for all $\alpha \in \mathbb{N}^N$. Notice that each measure $\mu \in \mathfrak{M}_N$ being finite is regular (e.g. see [22, Theorem 2.18]). As a consequence, every non-zero measure $\mu \in \mathfrak{M}_N$ has a non-empty closed support $\operatorname{supp} \mu$. Given $\mu \in \mathfrak{M}_N$, we define the linear functional $L_\mu : \mathcal{P}_N \longrightarrow \mathbb{C}$ via

$$L_\mu(p) \stackrel{\text{df}}{=} \int_{\mathbb{R}^N} p \, d\mu, \qquad p \in \mathcal{P}_N.$$

Call a linear functional L on $\mathcal{P}_N$ a *moment functional* [8] (*induced by a measure* $\mu \in \mathfrak{M}_N$) if $L = L_\mu$. Clearly every moment functional is positive definite. We say that a measure $\mu \in \mathfrak{M}_N$ *orthonormalizes* a sequence $\{Q_k\}_{k=0}^n$ $(0 \leqslant n \leqslant \infty)$ of column polynomials if $\{Q_k\}_{k=0}^n$ is L_μ-orthonormal. In this section we show that if there exists a measure orthonormalizing a sequence of column polynomials satisfying the three term recurrence relations modulo a $*$-ideal, then this ideal must be a set ideal. We first focus our interest on such ideals.

Let Δ be a subset of $\mathbb{R}^N$. Define the $*$-ideal $\mathcal{I}(\Delta)$ via

$$\mathcal{I}(\Delta) = \{p \in \mathcal{P}_N : p(x) = 0 \quad \text{for all } x \in \Delta\}.$$

We call $\mathcal{I}(\Delta)$ the *set ideal* (*induced by the set* Δ). If the interior of Δ is non-empty, then by the uniqueness theorem for polynomials $\mathcal{I}(\Delta) = \{0\}$. For $p \in \mathcal{P}_N$, we set $\mathcal{Z}_p \overset{\text{df}}{=} \{x \in \mathbb{R}^N : p(x) = 0\}$. To avoid ambiguity (e.g. $p = X_1$ can be regarded as a member of $\mathcal{P}_1$ as well as of $\mathcal{P}_2$) the number N appearing implicitly in the symbol $\mathcal{Z}_p$ will be always declared explicitly by writing $p \in \mathcal{P}_N$. Define

$$\overline{\Delta}^{\mathrm{z}} = \bigcap \left\{\mathcal{Z}_p : p \in \mathcal{P}_N \quad \text{and} \quad \Delta \subseteq \mathcal{Z}_p\right\}. \tag{46}$$

Notice that the set $\overline{\Delta}^{\mathrm{z}}$ remains unchanged if we replace $\mathcal{P}_N$ by $\mathcal{R}_N$ in (46) because for every $p \in \mathcal{P}_N$ the zero sets of polynomials p and $(\mathfrak{Re}\, p)^2 + (\mathfrak{Im}\, p)^2$ are equal to each other (cf. Section 1 for the definition of $\mathcal{R}_N$). Since each algebraic subset of $\mathbb{R}^N$ is of the form $\mathcal{Z}_p$ with some $p \in \mathcal{R}_N$, our definition of $\overline{\Delta}^{\mathrm{z}}$ coincides with the closure of Δ in the Zariski topology (which consists of complements of algebraic subsets of $\mathbb{R}^N$). Recall that $\mathbb{R}^N$ equipped with the Zariski topology is a topological T_1 space (because finite subsets of $\mathbb{R}^N$ are algebraic). As usual $\overline{\Delta}$ stands for the closure of Δ in the Euclidean topology of $\mathbb{R}^N$.

The reader can easily deduce from (46) that if $\Delta_1, \Delta_2 \subseteq \mathbb{R}^N$, then $\mathcal{I}(\Delta_1) \subseteq \mathcal{I}(\Delta_2)$ if and only if $\overline{\Delta}_2^{\mathrm{z}} \subseteq \overline{\Delta}_1^{\mathrm{z}}$, which in turn implies that

$$\mathcal{I}(\Delta_1) = \mathcal{I}(\Delta_2) \quad \text{if and only if} \quad \overline{\Delta}_1^{\mathrm{z}} = \overline{\Delta}_2^{\mathrm{z}}. \tag{47}$$

As a consequence, we get

$$\mathcal{I}(\Delta) = \mathcal{I}(\overline{\Delta}^{\mathrm{z}}), \quad \Delta \subseteq \mathbb{R}^N. \tag{48}$$

Moreover, since $\overline{\Delta}^{\mathrm{z}}$ is a real algebraic set, it is of the form $\mathcal{Z}_p$ with some $p \in \mathcal{P}_N$; therefore $\mathcal{I}(\Delta) = \mathcal{I}(\mathcal{Z}_p)$. In other words, every set ideal in $\mathcal{P}_N$ is of the form $\mathcal{I}(\mathcal{Z}_p)$ with some $p \in \mathcal{P}_N$. For fundamentals concerning algebraic sets and the Zariski topology we recommend the monographs [3,8].

The proof of the following fact is mainly included to keep the exposition as self-contained as possible.

Lemma 38. *If* $p \in \mathcal{P}_N$ *is such that* $\mathcal{Z}_p \neq \emptyset$, *then* $\mathcal{Z}_p$ *is finite if and only if* $\varkappa_{\mathcal{I}(\mathcal{Z}_p)} < \infty$ (*equivalently*: $\dim \mathcal{P}_N/\mathcal{I}(\mathcal{Z}_p) < \infty$). *If* $\operatorname{card} \mathcal{Z}_p < \infty$, *then*

$$\dim \mathcal{P}_N/\mathcal{I}(\mathcal{Z}_p) = \sum_{k=0}^{\varkappa_{\mathcal{I}(\mathcal{Z}_p)}} d_{\mathcal{I}(\mathcal{Z}_p)}(k) = \operatorname{card} \mathcal{Z}_p. \tag{49}$$

[8] In [10,11] "moment functional" is nothing but another name for linear functional on polynomials, which is not our case.

Proof. The equivalence $\varkappa_{\mathcal{I}(\mathcal{Z}_p)} < \infty \Longleftrightarrow \dim \mathcal{P}_N/\mathcal{I}(\mathcal{Z}_p) < \infty$ is clear. If the set $\mathcal{Z}_p$ is finite, then the (well defined) mapping

$$\mathcal{P}_N/\mathcal{I}(\mathcal{Z}_p) \ni r + \mathcal{I}(\mathcal{Z}_p) \longmapsto r|_{\mathcal{Z}_p} \in \mathbb{C}^{\mathcal{Z}_p}$$

is a linear isomorphism. This implies $\dim \mathcal{P}_N/\mathcal{I}(\mathcal{Z}_p) < \infty$ and (49).

Assume that $\dim \mathcal{P}_N/\mathcal{I}(\mathcal{Z}_p) < \infty$. Suppose that, contrary to our claim, the set $\mathcal{Z}_p$ is infinite. Take an arbitrary sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{Z}_p$ whose entries are pairwise distinct. For every integer $n \geqslant 1$, there exists $q_n \in \mathcal{P}_N$ with the property $q_n(x_j) = \delta_{n,j}$ for $j \in \overline{1,n}$. Then, as is easily seen, $q_1, q_2, \ldots$ are linearly $\mathcal{I}(\mathcal{Z}_p)$-independent, which contradicts $\dim \mathcal{P}_N/\mathcal{I}(\mathcal{Z}_p) < \infty$. □

We now describe the ideal of the form $\mathcal{V}_L$, where L is a moment functional.

Proposition 39. *If* $\mu \in \mathfrak{M}_N$ *and* $\mu \neq 0$, *then*

(i) $\mathcal{V}_{L_\mu} = \mathcal{I}(\operatorname{supp}\mu) = \mathcal{I}(\overline{\operatorname{supp}\mu}^{\mathrm{z}})$,
(ii) $\operatorname{supp}\mu$ *is finite if and only if* $\varkappa_{\mathcal{V}_{L_\mu}} < \infty$ (*equivalently*: $\dim \mathcal{P}_N/\mathcal{V}_{L_\mu} < \infty$).

Proof. (i) By (48), it is sufficient to show that $\mathcal{V}_{L_\mu} = \mathcal{I}(\operatorname{supp}\mu)$. Take $p \in \mathcal{P}_N$. The positive definiteness of L_μ, when combined with (43), implies that

$$\begin{aligned} p \in \mathcal{V}_{L_\mu} &\Longleftrightarrow L_\mu(pp^*) = 0 \\ &\Longleftrightarrow \int_{\mathbb{R}^N} |p(x)|^2 \,\mathrm{d}\mu(x) = 0 \\ &\Longleftrightarrow p = 0 \text{ a.e. } [\mu] \\ &\Longleftrightarrow \operatorname{supp}\mu \subseteq \mathcal{Z}_p \quad \text{(by the continuity of } p) \\ &\Longleftrightarrow p \in \mathcal{I}(\operatorname{supp}\mu). \end{aligned}$$

(ii) Notice first that $\overline{\operatorname{supp}\mu}^{\mathrm{z}} = \mathcal{Z}_p$ with some $p \in \mathcal{P}_N$, and consequently by (i) $\mathcal{V}_{L_\mu} = \mathcal{I}(\mathcal{Z}_p)$. Since $\operatorname{supp}\mu$ is finite if and only if $\overline{\operatorname{supp}\mu}^{\mathrm{z}}$ is finite, Lemma 38 completes the proof. □

In fact, Proposition 39 implies Lemma 38. Indeed, there exists a measure $\mu \in \mathfrak{M}_N$ such that $\operatorname{supp}\mu = \mathcal{Z}_p$ (see Lemma 40 below), and consequently, by part (i) of Proposition 39, $\mathcal{I}(\mathcal{Z}_p) = \mathcal{V}_{L_\mu}$. Applying part (ii) of Proposition 39 completes the proof of our claim.

It is well know that a moment functional L on $\mathcal{P}_N$ may be induced by more than one measure (cf. [4]). Nevertheless, if $\mu_1, \mu_2 \in \mathfrak{M}_N$ induce the same moment functional, then we deduce from (47) and Proposition 39 that $\overline{\operatorname{supp}\mu_1}^{\mathrm{z}} = \overline{\operatorname{supp}\mu_2}^{\mathrm{z}}$.

Lemma 40. *For any closed non-empty subset* Δ *of* $\mathbb{R}^N$, *there exists a probability measure* $\mu \in \mathfrak{M}_N$ *such that* $\operatorname{supp}\mu = \Delta$.

Proof. Assume Δ is infinite (the opposite case is trivial). Let $\{x_k\}_{k=1}^{\infty}$ be a sequence of distinct points, which is dense in Δ. Set $t_k = \max\{|x_{k,i}| : i = 1, \ldots, N\}$, where $x_k =$

$(x_{k,1}, \ldots, x_{k,N})$, and $\varepsilon_k = 2^{-k} \mathrm{e}^{-t_k}$ for $k \geqslant 1$. Then $\sum_{k=1}^{\infty} \varepsilon_k t_k^n < \infty$ for all $n \in \mathbb{N}$. Define ν via $\nu(\sigma) = \sum_{k: x_k \in \sigma} \varepsilon_k$ for a Borel subset σ of $\mathbb{R}^N$. Since

$$\int_{\mathbb{R}^N} |x^{\alpha}| \, \mathrm{d}\nu(x) \leqslant \sum_{k=1}^{\infty} \varepsilon_k t_k^{|\alpha|} < \infty, \quad \alpha \in \mathbb{N}^N,$$

the measure ν is in $\mathfrak{M}_N$ and supp $\nu = \Delta$. Hence $\mu \overset{\mathrm{df}}{=} \frac{1}{\nu(\mathbb{R}^N)} \nu$ is as desired. □

The following proposition shows that the problem of existence of an orthonormalizing measure can only be solved in the case of set ideals. It is worth noting that there are $*$-ideals which are not set ideals (e.g. $V = (X^s) \subseteq \mathcal{P}_1$, where $s \geqslant 2$).

Proposition 41. *Let V be a proper $*$-ideal in $\mathcal{P}_N$ and $\varnothing \neq \Delta \subseteq \mathbb{R}^N$.*

(i) *If a sequence $\{Q_k\}_{k=0}^{\varkappa_V}$ of real column polynomials (with $Q_0 = 1$) satisfies condition* (B) *of Theorem* 36, *and L defined by* (32) *is a moment functional induced by $\mu \in \mathfrak{M}_N$, then $V = \mathcal{I}(\text{supp}\, \mu)$.*

(ii) *If $V = \mathcal{I}(\Delta)$, then there exists a rigid V-basis $\{Q_k\}_{k=0}^{\varkappa_V}$ of $\mathcal{P}_N$ composed of real column polynomials (with $Q_0 = 1$), orthonormalized by some $\mu \in \mathfrak{M}_N$ and satisfying condition* (B) *of Theorem* 36.

Proof. (i) By Theorem 36 and part (i) of Proposition 39, $V = \mathcal{V}_L = \mathcal{I}(\text{supp}\, \mu)$.

(ii) In virtue of Lemma 40, $\overline{\Delta} = \text{supp}\, \mu$ for some $\mu \in \mathfrak{M}_N$. Applying Proposition 32 to L_μ (see also Remark 33), we find an L_μ-orthonormal sequence $\{Q_k\}_{k=0}^{\varkappa_{\mathcal{V}_{L_\mu}}}$ of real column polynomials (with $Q_0 = 1$), which is a rigid $\mathcal{V}_{L_\mu}$-basis of $\mathcal{P}_N$. By part (i) of Proposition 39, $\mathcal{V}_{L_\mu} = \mathcal{I}(\Delta)$. This and Theorem 36 complete the proof. □

10. Existence of orthogonalizing measures: general approach

In this section we distinguish the class of $*$-ideals V for which every sequence of real column polynomials satisfying the three term recurrence relations modulo V is orthonormalized by a measure; we will call them $*$-ideals of type C. As will be shown below, type C is closely connected with the notions of types A and B introduced in [24], namely, $*$-ideals of the form $\mathcal{I}(\mathcal{Z}_p)$, where $p \in \mathcal{P}_N$ is an arbitrary polynomial of type A or B, are always of type C. This crucial observation motivates our interest in types A and B which additionally are easier to deal with than type C. On the other hand, Proposition 41 guarantees that all (but some pathological cases) $*$-ideals of type C must be of the form $\mathcal{I}(\mathcal{Z}_p)$ with some $p \in \mathcal{P}_N$.

Given a (complex) inner product space $\mathcal{D}$, we denote by $\boldsymbol{L}^{\#}(\mathcal{D})$ the algebra of all linear operators $A : \mathcal{D} \to \mathcal{D}$ for which there exists a linear operator $A^{\#} : \mathcal{D} \to \mathcal{D}$ such that $\langle Af, g \rangle = \langle f, A^{\#} g \rangle$ for all $f, g \in \mathcal{D}$; such $A^{\#}$ is unique and the mapping $\boldsymbol{L}^{\#}(\mathcal{D}) \ni A \longmapsto A^{\#} \in \boldsymbol{L}^{\#}(\mathcal{D})$ is the involution in $\boldsymbol{L}^{\#}(\mathcal{D})$. The identity operator on $\mathcal{D}$ is denoted by $I_{\mathcal{D}}$ (or simply I). Set $\boldsymbol{L}^{\#}_{\mathrm{s}}(\mathcal{D}) = \{A \in \boldsymbol{L}^{\#}(\mathcal{D}) : A = A^{\#}\}$. An N-tuple $S = (S_1, \ldots, S_N) \in \boldsymbol{L}^{\#}_{\mathrm{s}}(\mathcal{D})^N$

is said to be *commuting* if all the operators $S_1, \ldots, S_N$ mutually commute. For such N-tuple S and a polynomial $r = \sum_{\alpha \in \mathbb{N}^N} a_\alpha X^\alpha$ with complex coefficients a_α (vanishing for all but a finite number of indices α), we define the operator $r(S) \in \boldsymbol{L}^\#(\mathcal{D})$ via

$$r(S) = \sum_{\alpha \in \mathbb{N}^N} a_\alpha S^\alpha \quad \text{with} \quad S^\alpha = S_1^{\alpha_1} \ldots S_N^{\alpha_N} \quad (S^0 = I_{\mathcal{D}}).$$

We say that a commuting N-tuple $S \in \boldsymbol{L}_\mathrm{s}^\#(\mathcal{D})^N$ is *cyclic* if there exists a vector $e \in \mathcal{D}$ (called a *cyclic vector* of S) such that $\mathcal{D} = \{r(S)e : r \in \mathcal{P}_N\}$.

A polynomial $p \in \mathcal{P}_N$ is said to be of *type* A$^\mathrm{o}$ (resp. of *type* A), if for every inner product space $\mathcal{D}$ and for every commuting N-tuple (resp. cyclic commuting N-tuple) $S = (S_1, \ldots, S_N) \in \boldsymbol{L}_\mathrm{s}^\#(\mathcal{D})^N$ satisfying $p(S) = 0$, there exists an N-tuple $(T_1, \ldots, T_N)$ of spectrally commuting self-adjoint operators [9] in a Hilbert space $\mathcal{K} \supseteq \mathcal{D}$ (isometric embedding) such that $S_j \subseteq T_j$ for all $j = 1, \ldots, N$. It is a matter of direct verification that a polynomial $p \in \mathcal{P}_N$ is of type A (resp. A$^\mathrm{o}$) if and only if the polynomial $(\mathfrak{Re}\, p)^2 + (\mathfrak{Im}\, p)^2 \in \mathcal{R}_N$ is of type A (resp. A$^\mathrm{o}$).

We say that $p \in \mathcal{P}_N$ is of *type* B, if every positive definite linear functional $L : \mathcal{P}_N \to \mathbb{C}$ vanishing on $\mathcal{I}(\mathcal{Z}_p)$ is a moment functional.

A $*$-ideal V in $\mathcal{P}_N$ is said to be of *type* C, if either $V = \mathcal{P}_N$ or $V \neq \mathcal{P}_N$ and for every sequence $\{Q_k\}_{k=0}^{\varkappa_V}$ of real column polynomials satisfying condition (B) of Theorem 36 with $Q_0 = 1$, the linear functional L defined by (32) is a moment functional. A polynomial $p \in \mathcal{P}_N$ is of *type* C if the set ideal $\mathcal{I}(\mathcal{Z}_p)$ is of type C.

When we consider types A$^\mathrm{o}$ and A in the case of a specific polynomial we have to declare the number of its indeterminates in advance. The same refers to types B and C (in the latter case only the zero ideal needs a declaration). The dependence of types A$^\mathrm{o}$ and A on the number of indeterminates is illustrated below.

The zero polynomial is of type A$^\mathrm{o}$ as a member of $\mathcal{P}_1$ and is not of type A as a member of $\mathcal{P}_2$. (50)

The first statement of (50) is an immediate consequence of the well known fact asserting that every symmetric operator in a Hilbert space has a self-adjoint extension possibly in a larger Hilbert space (cf. [1, §111 Theorem 1] and [25, Proposition 1]), while the other can be deduced from [4, Theorem 6.3.4] via [26, Proposition 2]. In view of Proposition 42, the Hamburger theorem (cf. [4]) just amounts to saying that the zero polynomial is of type A as a member of $\mathcal{P}_1$.

Proposition 42. *A polynomial $p \in \mathcal{P}_N$ is of type A if and only if every positive definite linear functional $L : \mathcal{P}_N \to \mathbb{C}$ vanishing on the principal ideal (p) is a moment functional. A $*$-ideal V in $\mathcal{P}_N$ is of type C if and only if every positive definite linear functional $L : \mathcal{P}_N \to \mathbb{C}$ satisfying $\mathcal{V}_L = V$ is a moment functional. If $p \in \mathcal{P}_N$ is of type A* (*resp. B*), *then it is of type B* (*resp. C*).*

Following [27, Section 18] it is possible to characterize type A$^\mathrm{o}$ similarly to type A replacing functionals L by mappings taking values in sesquilinear forms.

[9] That is the spectral measures of the operators $T_1, \ldots, T_N$ commute.

Proof of Proposition 42. The proof of the first equivalence (concerning type A) is essentially the same as that of [26, Proposition 2] (notice that a linear functional $L : \mathcal{P}_N \to \mathbb{C}$ is positive definite and $L|_{(p)} = 0$ if and only if the corresponding multisequence $c = \{L(X^\alpha)\}_{\alpha \in \mathbb{N}^N}$ is positive definite and $c_p = p(E)c = 0$, where $E = (E_1, \ldots, E_N)$ is the N-tuple of linear operators acting on $\mathbb{C}^{\mathbb{N}^N}$ via $E_j(c)(\alpha) = c(\alpha + e_j)$ with $e_j \stackrel{\mathrm{df}}{=} (\delta_{1,j}, \ldots, \delta_{N,j})$. The other equivalence (concerning type C) can be deduced from Proposition 32, Remark 33 and Theorem 36. The last statement is evident, because $(p) \subseteq \mathcal{I}(\mathcal{Z}_p)$ and $\mathcal{V}_L \subseteq \ker L$. □

In view of Proposition 42 and Theorem 43, as far as types A$^{\mathrm{o}}$, A, B and C are concerned, only the polynomials with unbounded zero sets need be investigated. It is worth noting that the proof of Theorem 43 refers to the positivstellensatz (cf. [8, Corollaire 4.4.3]).

Theorem 43. *Every polynomial $p \in \mathcal{P}_N$ with compact* [10] *$\mathcal{Z}_p$ is of type* A$^{\mathrm{o}}$, *and, in consequence, of type C.*

Proof. Replacing, if necessary, p by $(\mathfrak{Re}\, p)^2 + (\mathfrak{Im}\, p)^2$, we can limit ourselves to the case $p \in \mathcal{R}_N$. Proposition 2 of [26] adapted to our setting guarantees that each $p \in \mathcal{R}_N$ of type A with compact $\mathcal{Z}_p$ is of type A$^{\mathrm{o}}$. On the other hand, Theorem 1 of [23] applied to $R = \{p, -p\}$ and combined with Proposition 42 implies that each $p \in \mathcal{R}_N$ with compact $\mathcal{Z}_p$ is of type A (see the proof of Proposition 42 for a hint). Applying Proposition 42 completes the proof. □

Question 1. Does there exist a polynomial $p \in \mathcal{P}_N$ of type A which is not of type A$^{\mathrm{o}}$?

The interested reader is referred to papers [7,12] which are in a way related to Question 1. It follows from Proposition 42 that the definitions of types A and B coincide for every $p \in \mathcal{P}_N$ for which $\mathcal{I}(\mathcal{Z}_p) = (p)$.

Question 2. Do types A and B (resp. B and C) coincide for every $p \in \mathcal{P}_N$?

Remark 44. Let us discuss in more detail the definitions of types A and C.

(a) If $p \in \mathcal{P}_N$ is of type A, then the set $\boldsymbol{L}_+(p)$ of all positive definite linear functionals on $\mathcal{P}_N$ vanishing on (p) coincides with the set $\boldsymbol{L}_{\mathrm{m}}(p)$ of all moment functionals on $\mathcal{P}_N$ induced by measures supported in $\mathcal{Z}_p$ (both sets are non-empty).

Indeed, if $L \in \boldsymbol{L}_+(p)$, then by Proposition 42 L is a moment functional induced by a measure $\mu \in \mathfrak{M}_N$. Since $L|_{(p)} = 0$, we get $0 = L(p^* p) = \int_{\mathbb{R}^N} |p(x)|^2 \,\mathrm{d}\mu(x)$, which implies that the closed support $\operatorname{supp} \mu$ of μ is contained in $\mathcal{Z}_p$. Thus $L \in \boldsymbol{L}_{\mathrm{m}}(p)$. The converse implication is plain.

(b) If $p \in \mathcal{P}_N$ is of type A, then $\{\mathcal{V}_L : L \in \boldsymbol{L}_+(p)\} = \{\mathcal{I}(\Delta) : \Delta \subseteq \mathcal{Z}_p\}$.

[10] The case $\mathcal{Z}_p = \varnothing$ is not excluded.

Indeed, if $L \in \boldsymbol{L}_+(p)$, then by (a), $L = L_\mu$ with some $\mu \in \mathfrak{M}_N$ supported in $\mathcal{Z}_p$. As a consequence, by part (i) of Proposition 39, we have $\mathcal{V}_L = \mathcal{I}(\varDelta)$ with $\varDelta \overset{\text{df}}{=} \operatorname{supp}\mu \subseteq \mathcal{Z}_p$. Conversely, if $\varDelta$ is a subset of $\mathcal{Z}_p$, then by Lemma 40 there exists $\mu \in \mathfrak{M}_N$ such that $\operatorname{supp}\mu = \overline{\varDelta}$. Then $L_\mu \in \boldsymbol{L}_+(p)$ and, by part (i) of Proposition 39, $\mathcal{I}(\varDelta) = \mathcal{I}(\overline{\varDelta}) = \mathcal{V}_{L_\mu}$ (the case $\varDelta = \varnothing$ is trivial).

(c) Suppose that a $*$-ideal V in $\mathcal{P}_N$ is of type C. Denote by $\boldsymbol{L}'_+(V)$ the set of all positive definite linear functionals L on $\mathcal{P}_N$ such that $\mathcal{V}_L = V$. If $\boldsymbol{L}'_+(V) \neq \varnothing$, then V is a set ideal. If V is a set ideal induced by $\varDelta \subseteq \mathbb{R}^N$, then $\boldsymbol{L}'_+(V)$ is equal to the set $\boldsymbol{L}'_{\mathrm{m}}(V)$ of all moment functionals L on $\mathcal{P}_N$ induced by measures μ such that $\overline{\varDelta}^{\mathrm{z}} = \overline{\operatorname{supp}\mu}^{\mathrm{z}}$; moreover then $\boldsymbol{L}'_+(V) \neq \varnothing$.

Indeed, if $\boldsymbol{L}'_+(V) \neq \varnothing$, then by Propositions 42 and 39, V is a set ideal. If $V = \mathcal{I}(\varDelta)$ for some $\varDelta \subseteq \mathbb{R}^N$, then the equality $\boldsymbol{L}'_+(V) = \boldsymbol{L}'_{\mathrm{m}}(V)$ can be inferred from (47), (48) and Propositions 42 and 39, while $\boldsymbol{L}'_{\mathrm{m}}(V) \neq \varnothing$ from Lemma 40.

Example 45. Consider the $*$-ideal $V = (X^s)$ in $\mathcal{P}_1$ with $s \geqslant 2$. We show that $\boldsymbol{L}'_+(V) = \varnothing$. In the contrary case, there exists $L \in \boldsymbol{L}'_+(V)$. Since $V = \mathcal{V}_L \subseteq \ker L$, we get $L(X^j) = 0$ for all $j \geqslant s$. This and the Cauchy–Schwarz inequality lead to

$$0 \leqslant |L(X^{s-1})|^2 \leqslant L(1)L(X^{2(s-1)}) = 0 \quad (\text{because } 2(s-1) \geqslant s).$$

Hence $L(X^j) = 0$ for all $j \geqslant s-1$. By backward induction, we get $L(X^i) = 0$ for all $i \geqslant 1$. This implies that $L(r) = L(1)r(0)$ for $r \in \mathcal{P}_1$. If $L(1) = 0$, then $\mathcal{V}_L = \mathcal{P}_1 \neq V$, a contradiction. If $L(1) \neq 0$, then $\mathcal{V}_L = (X) \neq V$, a contradiction. Therefore $\boldsymbol{L}'_+(V) = \varnothing$, which by Theorem 36 implies that there exists no sequence $\{Q_k\}_{k=0}^{\varkappa_V}$ of real column polynomials (with $Q_0 = 1$) satisfying condition (B) of Theorem 36 (here $\varkappa_V = s-1$ and $\Sigma_k^V = X^k$ for $k = 0, \ldots, s-1$). By Proposition 42, V is of type C but evidently it is not a set ideal. Summarizing, we see that part (ii) of Proposition 41 is no longer true for $*$-ideals which are not set ideals.

Let us list some properties of types A$^{\mathrm{o}}$, A, B and C. Our first goal is to discuss whether a polynomial $p \in \mathcal{P}_N$ of type A (resp. A$^{\mathrm{o}}$, B, C) composed with a polynomial mapping $\varphi : \mathbb{R}^M \to \mathbb{R}^N$ is still of the same type. In general, this procedure does not preserve types A$^{\mathrm{o}}$, A and B. To see an example, define $p = X_2 - X_1^2 \in \mathcal{P}_2$ and $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ via $\varphi(x_1, x_2) = (x_1, x_1^2)$ for $x_1, x_2 \in \mathbb{R}$. Then p is of type A$^{\mathrm{o}}$ (because every symmetric operator has a self-adjoint extension possibly in a larger Hilbert space; see also the proof of [26, Proposition 3]), whereas $p \circ \varphi = 0 \in \mathcal{P}_2$ is not of type B (cf. [4, Theorem 6.3.4]). On the other hand, types A$^{\mathrm{o}}$, A, B and C are preserved by polynomial automorphisms (see [29] for fundamentals of the theory of polynomial automorphisms). Prototypes of this property have already been indicated in [24,27].

Proposition 46. *Assume that $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ is a polynomial automorphism. If $p \in \mathcal{P}_N$ is of type A* (*resp.* A$^{\mathrm{o}}$, *B*), *then $p \circ \varphi \in \mathcal{P}_N$ is of type A* (*resp.* A$^{\mathrm{o}}$, *B*). *If V is a $*$-ideal in $\mathcal{P}_N$ of type C, then so is the $*$-ideal $V_\varphi \overset{\text{df}}{=} \{q \circ \varphi : q \in V\}$ in $\mathcal{P}_N$.*

Proof. Throughout the proof we use the following notation $\varphi = (\varphi_1, \ldots, \varphi_N)$ and $\varphi^{-1} = (\psi_1, \ldots, \psi_N)$, where $\varphi_1, \ldots, \varphi_N, \psi_1, \ldots, \psi_N \in \mathcal{R}_N$. Denote by C_φ the $*$-algebra isomorphism of $\mathcal{P}_N$ given by $C_\varphi(q) = q \circ \varphi$ for $q \in \mathcal{P}_N$.

Suppose $p \in \mathcal{P}_N$ is of type A^{o}. Take a commuting N-tuple $S = (S_1, \ldots, S_N) \in \boldsymbol{L}_{\mathrm{s}}^{\#}(\mathcal{D})^N$ such that $p(\varphi(S)) = 0$. Then there exists an N-tuple $T = (T_1, \ldots, T_N)$ of spectrally commuting self-adjoint operators in a Hilbert space $\mathcal{K} \supseteq \mathcal{D}$ such that $\varphi_j(S) \subseteq T_j$ for all $j = 1, \ldots, N$. Let E be the joint spectral measure of T (cf. [5]). Then the N-tuple $\left(\int_{\mathbb{R}^N} \psi_1 \, \mathrm{d}E, \ldots, \int_{\mathbb{R}^N} \psi_N \, \mathrm{d}E\right)$ is composed of spectrally commuting self-adjoint operators and $S_j = \psi_j(\varphi(S)) \subseteq \psi_j(T) \subseteq \int_{\mathbb{R}^N} \psi_j \, \mathrm{d}E$ for every $j = 1, \ldots, N$. This shows that $p \circ \varphi$ is of type A^{o}. The case of type A can be handled in much the same way, because S is cyclic if and only if $\varphi(S)$ is cycle (with the same cyclic vector).

Proposition 42 helps us to establish the case of type C. Indeed, if $L : \mathcal{P}_N \to \mathbb{C}$ is a positive definite linear functional such that $\mathcal{V}_L = V_\varphi$, then $L \circ C_\varphi$ is a positive definite linear functional as well, and $\mathcal{V}_{L \circ C_\varphi} = C_\varphi^{-1}(\mathcal{V}_L) = C_\varphi^{-1}(C_\varphi(V)) = V$. Since V is of type C, there exists $\mu \in \mathfrak{M}_N$ such that $L \circ C_\varphi = L_\mu$. Applying the measure transport theorem (cf. [16, Theorem C, p. 163]) we get $L = L_{\mu \circ \varphi}$, which proves our claim. A similar argument combined with the equality $\mathcal{I}(\mathcal{Z}_{p \circ \varphi}) = C_\varphi(\mathcal{I}(\mathcal{Z}_p))$ settles the case of type B. □

Continuing our discussion, we show that "freezing variables" preserves type A.

Proposition 47. *Let $N \geqslant 2$, $k \in \{1, \ldots, N-1\}$ and $\lambda_{k+1}, \ldots, \lambda_N \in \mathbb{R}$. If $p \in \mathcal{P}_N$ is of type A (resp. A^{o}), then so is $p(X_1, \ldots, X_k, \lambda_{k+1}, \ldots, \lambda_N) \in \mathcal{P}_k$.*

Proof. If $S = (S_1, \ldots, S_k) \in \boldsymbol{L}_{\mathrm{s}}^{\#}(\mathcal{D})^k$ is a commuting k-tuple such that

$$p(S_1, \ldots, S_k, \lambda_{k+1}, \ldots, \lambda_N) = 0,$$

then $\widetilde{S} \stackrel{\mathrm{df}}{=} (S_1, \ldots, S_k, \lambda_{k+1} I_{\mathcal{D}}, \ldots, \lambda_N I_{\mathcal{D}}) \in \boldsymbol{L}_{\mathrm{s}}^{\#}(\mathcal{D})^N$ is a commuting N-tuple such that $p(\widetilde{S}) = 0$. Moreover, every cyclic vector of S is a cyclic vector of $\widetilde{S}$. These two facts enable us to complete the proof. □

Regarding Proposition 47, we see that the polynomial $p = X_2 \in \mathcal{P}_2$ is of type A^{o}, $q \stackrel{\mathrm{df}}{=} p(X_1, 0) \in \mathcal{P}_1$ is of type A^{o} (and $\mathcal{Z}_q = \mathbb{R}$), whereas q as a member of $\mathcal{P}_2$ is not of type A (cf. (50)). This means that it is essential in Proposition 47 to treat $p(X_1, \ldots, X_k, \lambda_{k+1}, \ldots, \lambda_N)$ as a polynomial in k indeterminates. On the other hand, freezing variables may lead to $\mathcal{Z}_q = \varnothing$, e.g. the polynomial $p = X_1^2 + X_2^2 - 1 \in \mathcal{P}_2$ is of type A^{o},[11] hence $q \stackrel{\mathrm{df}}{=} p(X_1, \lambda_2) \in \mathcal{P}_1$ is of type A^{o} for every $\lambda_2 \in \mathbb{R}$, though $\mathcal{Z}_q = \varnothing$ if $\lambda_2^2 > 1$.

The question arises whether the algebraic subsets or supersets of $\mathcal{Z}_p$, where $p \in \mathcal{P}_N$ is a polynomial of type A (resp. A^{o}, B, C), are still the zero sets of polynomials of the same

[11] This follows either from Theorem 43 or from the ensuing simple reasoning: if $S = (S_1, S_2) \in \boldsymbol{L}_{\mathrm{s}}^{\#}(\mathcal{D})^2$ is a commuting pair such that $S_1^2 + S_2^2 = I$, then $\|S_j f\|^2 \leqslant \langle (S_1^2 + S_2^2) f, f \rangle = \|f\|^2$ for $f \in \mathcal{D}$ and $j = 1, 2$, which implies that the closures of S_1 and S_2, considered as operators in the Hilbert space completion of $\mathcal{D}$, are commuting bounded self-adjoint operators.

type. According to [24, Theorem 6.3] (see also Example 54, in which a particular case of [24, Theorem 6.3] is explicitly quoted), one can find two polynomials $q, r \in \mathcal{P}_N$, both of type A, such that $\varnothing \subsetneq \mathcal{Z}_q \subsetneq \mathcal{Z}_p$, where $p \stackrel{\text{df}}{=} qr$ is not of type B. On the contrary, as shown below in Lemma 49, the zero set of a polynomial of type A enlarged by an arbitrary finite set is still the zero set of a polynomial of type A. An answer to the question of diminishing algebraic sets is given in Proposition 48, which generalizes [27, Proposition 55] to the case of type A$^{\text{o}}$ and provides a hereditary property of type B.

Proposition 48. *Let $p, q \in \mathcal{P}_N$ be such that $\mathcal{Z}_q \subseteq \mathcal{Z}_p$. If q divides p and p is of type A* (*resp.* A$^{\text{o}}$), *then so is q. If p is of type B, then so is q. If p is of type A, then every $*$-ideal V in $\mathcal{P}_N$ which contains (p) is of type C. In particular, if p is of type A, then q is of type C.*

Proof. Suppose p is of type A$^{\text{o}}$ and $p = qr$ with some $r \in \mathcal{P}_N$. If $S \in \boldsymbol{L}^{\#}_{\text{s}}(\mathcal{D})^N$ is a commuting N-tuple such that $q(S) = 0$, then $p(S) = q(S)r(S) = 0$ and hence q is of type A$^{\text{o}}$. The same applies to type A.

Suppose now that p is of type B. Let L be a positive definite linear functional on $\mathcal{P}_N$ vanishing on $\mathcal{I}(\mathcal{Z}_q)$. Since $\mathcal{I}(\mathcal{Z}_p) \subseteq \mathcal{I}(\mathcal{Z}_q)$, L must be a moment functional. Hence q is of type B.

Finally, let us assume that p is of type A. Suppose L is a positive definite linear functional on $\mathcal{P}_N$ such that $\mathcal{V}_L = V$. Since $(p) \subseteq V = \mathcal{V}_L \subseteq \ker L$, we infer from Proposition 42 that L is a moment functional. Applying once more Proposition 42, we conclude that V is of type C. This and the inclusions $(p) \subseteq \mathcal{I}(\mathcal{Z}_p) \subseteq \mathcal{I}(\mathcal{Z}_q)$ imply q is of type C. $\square$

To illustrate Proposition 48 set

$$p = X_1^2 + X_2^2 + X_3^2 - 1 \quad \text{and} \quad q = (X_1^2 + X_2^2 - 1)^2 + X_3^2.$$

The polynomials p and q are of type A$^{\text{o}}$ as members of $\mathcal{P}_3$ (compare with footnote 11), and $\varnothing \subsetneq \mathcal{Z}_q \subsetneq \mathcal{Z}_p$. Moreover, the polynomial p is irreducible in $\mathcal{P}_3$ (consequently the zero sets of divisors of p are of the form $\varnothing$ and $\mathcal{Z}_p$) and q is irreducible in $\mathcal{R}_3$ (however $q = (X_1^2 + X_2^2 - 1 + \mathrm{i}X_3)(X_1^2 + X_2^2 - 1 - \mathrm{i}X_3)$). In virtue of [8, Théorème 4.5.1], we see that $\mathcal{I}(\mathcal{Z}_p) = (p)$ and $\mathcal{I}(\mathcal{Z}_{X_1^2+X_2^2-1}) = (X_1^2 + X_2^2 - 1) (\subseteq \mathcal{P}_2)$. The latter equality turns out to be useful for proving that $\mathcal{I}(\mathcal{Z}_q) = (X_1^2 + X_2^2 - 1, X_3)$.

Given $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$, we define the polynomial $w_a = \sum_{k=1}^N (X_k - a_k)^2$. It is clear that $\mathcal{Z}_{w_a} = \{a\}$.

Lemma 49. *Let $a^{(1)}, \ldots, a^{(n)} \in \mathbb{R}^N$ and $q \stackrel{\text{df}}{=} \prod_{j=1}^n w_{a^{(j)}}$. Then a polynomial $p \in \mathcal{P}_N$ is of type A* (*resp.* A$^{\text{o}}$) *if and only if $pq \in \mathcal{P}_N$ is of type A* (*resp.* A$^{\text{o}}$).

Proof. By induction on n and by the following property (cf. Proposition 46)

> if $b = (b_1, \ldots, b_N) \in \mathbb{R}^N$, then a polynomial $r \in \mathcal{P}_N$ is of type A (resp. A$^{\text{o}}$) if and only if $r(X_1 + b_1, \ldots, X_N + b_N) \in \mathcal{P}_N$ is of type A (resp. A$^{\text{o}}$),

we are reduced to showing that $p \in \mathcal{P}_N$ is of type A (resp. A$^{\mathrm{o}}$) if and only if $pw_0 \in \mathcal{P}_N$ is of type A (resp. A$^{\mathrm{o}}$).

Suppose $p \in \mathcal{P}_N$ is of type A$^{\mathrm{o}}$. Let $S = (S_1, \ldots, S_N) \in \boldsymbol{L}_{\mathrm{s}}^{\#}(\mathcal{D})^N$ be a commuting N-tuple such that $(pw_0)(S) = 0$, i.e.

$$p(S)(S_1^2 + \cdots + S_N^2) = 0. \tag{51}$$

Multiplying both sides of (51) by $p(S)^{\#}$ we get

$$(p(S)S_1)^{\#}(p(S)S_1) + \cdots + (p(S)S_N)^{\#}(p(S)S_N) = 0.$$

This in turn implies that

$$p(S)S_k = 0, \quad k = 1, \ldots, N. \tag{52}$$

If $r \in \mathcal{P}_N$, then there exist $r_1, \ldots, r_N \in \mathcal{P}_N$ such that $r = r(0) + \sum_{k=1}^N X_k r_k$. Hence, by (52), we have $r(S)p(S) = r(0)p(S)$. Substituting $r = p^*$ leads to

$$p(S)^{\#} p(S) = \overline{p(0)}\, p(S). \tag{53}$$

If $p(0) = 0$, then, by (53), $0 = p(S)^{\#} p(S)$ and consequently $p(S) = 0$. Since p is of type A$^{\mathrm{o}}$, we arrive at the desired conclusion. On the other hand, if $p(0) \neq 0$, then we can assume without loss of generality that $p(0) = 1$. This, when combined with (53), implies $Q^{\#} = Q = Q^2$, where $Q \overset{\mathrm{df}}{=} p(S)$. It is now a matter of routine to show that $\mathcal{D} = \mathcal{N}(Q) \oplus Q(\mathcal{D})$, where $\mathcal{N}(Q) = \{f \in \mathcal{D} : Q(f) = 0\}$. Since $S_k Q = Q S_k$ for all $k = 1, \ldots, N$, we conclude that both $\mathcal{N}(Q)$ and $Q(\mathcal{D})$ are invariant for each S_k and consequently

$$S_k = S_k|_{\mathcal{N}(Q)} \oplus S_k|_{Q(\mathcal{D})}, \quad k = 1, \ldots, N. \tag{54}$$

Since p is of type A$^{\mathrm{o}}$ and $p(S_1|_{\mathcal{N}(Q)}, \ldots, S_N|_{\mathcal{N}(Q)}) = 0$, there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{N}(Q)$ and N-tuple $(T_1, \ldots, T_N)$ of spectrally commuting self-adjoint operators in $\mathcal{K}$ such that $S_k|_{\mathcal{N}(Q)} \subseteq T_k$ for $k = 1, \ldots, N$. It follows from (52) that $S_k|_{Q(\mathcal{D})} = 0$ for all $k = 1, \ldots, N$. Hence $(T_1 \oplus 0, \ldots, T_N \oplus 0)$ is an N-tuple of spectrally commuting self-adjoint operators, which by (54) extends S; here 0 is understood as the zero operator defined on the Hilbert space completion of $Q(\mathcal{D})$.

To make the above proof valid for type A, we only have to show that if S is cyclic, then so is $S|_{\mathcal{N}(Q)} \overset{\mathrm{df}}{=} (S_1|_{\mathcal{N}(Q)}, \ldots, S_N|_{\mathcal{N}(Q)})$. Indeed, if e is a cyclic vector of S, then

$$\begin{aligned}\{r(S)(I_{\mathcal{D}} - Q)e : r \in \mathcal{P}_N\} &= \{(I_{\mathcal{D}} - Q)r(S)e : r \in \mathcal{P}_N\} \\ &= (I_{\mathcal{D}} - Q)(\mathcal{D}) = \mathcal{N}(Q),\end{aligned}$$

which means that $(I_{\mathcal{D}} - Q)e$ is a cyclic vector of $S|_{\mathcal{N}(Q)}$.

The “if” part of the conclusion follows from Proposition 48. □

Substituting the polynomial $p = 1$ (which is of type A$^{\mathrm{o}}$ as a member of $\mathcal{P}_N$) into Lemma 49, we see that $\prod_{j=1}^n w_{a^{(j)}} \in \mathcal{P}_N$ is of type A$^{\mathrm{o}}$. Making use of the property stated in Lemma 49, we prove a similar feature of set ideals of type C.

Theorem 50. *Assume that $p \in \mathcal{P}_N$ is of type A and $\varDelta$ is a finite subset of $\mathbb{R}^N$. Then the $*$-ideal $\mathcal{I}(\mathcal{Z}_p \cup \varDelta)$ in $\mathcal{P}_N$ is of type C. In particular, $*$-ideals $\mathcal{I}(\mathcal{Z}_p)$ and $\mathcal{I}(\varDelta)$ in $\mathcal{P}_N$ are of type C.*

Proof. If $\varDelta = \varnothing$, then it suffices to apply Proposition 42. Suppose now that $\varDelta = \{a^{(1)}, \dots, a^{(n)}\}$. Then clearly $\mathcal{Z}_p \cup \varDelta = \mathcal{Z}_{pq}$, where $q \overset{\mathrm{df}}{=} \prod_{j=1}^n w_{a^{(j)}}$. This, when combined with Lemma 49 and Proposition 42, implies that the $*$-ideal $\mathcal{I}(\mathcal{Z}_p \cup \varDelta)$ in $\mathcal{P}_N$ is of type C. Substituting $p = 1$ yields the $*$-ideal $\mathcal{I}(\varDelta)$ in $\mathcal{P}_N$ is of type C. □

Remark 51. We show independently of Theorems 43 and 50 that for any finite subset $\varDelta$ of $\mathbb{R}^N$, the $*$-ideal $V \overset{\mathrm{df}}{=} \mathcal{I}(\varDelta)$ in $\mathcal{P}_N$ is of type C. By Lemma 38, $\dim \mathcal{P}_N/V < \infty$. Let L be a positive definite linear functional on $\mathcal{P}_N$ such that $\mathcal{V}_L = V$. Plainly,

$$\mathcal{P}_N/V \times \mathcal{P}_N/V \ni (q+V, r+V) \longmapsto \langle q+V, r+V\rangle_L \overset{\mathrm{df}}{=} L(qr^*) \in \mathbb{C}$$

is a well defined inner product in $\mathcal{P}_N/V$. Define the multiplication operators $S_1, \dots, S_N$ on $\mathcal{P}_N/V$ via $S_j(q+V) = X_j q + V$ for $q \in \mathcal{P}_N$ and $j = 1, \dots, N$. Then $S = (S_1, \dots, S_N)$ is a commuting N-tuple of bounded self-adjoint operators on a finite-dimensional Hilbert space $\mathcal{P}_N/V$. Let E be the joint spectral measure of S and set $\mu(\cdot) = \langle E(\cdot)(X^0+V), X^0+V\rangle_L$. Then

$$L(q) = \langle q+V, X^0+V\rangle_L = \langle q(S)(X^0+V), X^0+V\rangle_L = \int_{\mathbb{R}^N} q \, \mathrm{d}\mu \tag{55}$$

for every $q \in \mathcal{P}_N$, which completes the proof.

To get the feeling of the operator theory approach promoted in our paper, let us discuss the positive polynomial $p = X_1^2 X_2^2 (X_1^2 + X_2^2 - 1) + 1 \in \mathcal{P}_2$, which is not a sum of squares of real polynomials (cf. [4, Lemma 6.3.1]; see also [2] for an affirmative answer to the related Hilbert's 17th problem). We show independently of Theorem 43 that p is of type A^{o}. Take a commuting pair $S = (S_1, S_2) \in \boldsymbol{L}_{\mathrm{s}}^{\#}(\mathcal{D})^2$ such that

$$S_1^2 S_2^2 (I - S_1^2 - S_2^2) = I. \tag{56}$$

Then clearly S_1 and S_2 are bijections. It follows from (56) that

$$\|S_1 S_2 h\|^2 = \|h\|^2 + \|S_1^2 S_2 h\|^2 + \|S_2^2 S_1 h\|^2, \quad h \in \mathcal{D}. \tag{57}$$

Consequently, we have

$$\|S_1 S_2 h\| \geqslant \|h\|, \quad h \in \mathcal{D}. \tag{58}$$

Applying once more (57), we get $\|S_j(S_1 S_2 h)\| \leqslant \|S_1 S_2 h\|$ for $h \in \mathcal{D}$ and $j = 1, 2$. Since $S_1 S_2$ is a bijection, we conclude that $\|S_j h\| \leqslant \|h\|$ for $h \in \mathcal{D}$ and $j = 1, 2$. Hence $\|S_1 S_2 h\| \leqslant \|h\|$ for $h \in \mathcal{D}$. This and (58) lead to $\|S_1 S_2 h\| = \|h\|$ for $h \in \mathcal{D}$. The last equality and (57) imply $S_1^2 S_2 = S_2^2 S_1 = 0$. However S_1 and S_2 are bijections, and so $\mathcal{D} = \{0\}$, which completes the proof of our claim.

11. Existence of orthogonalizing measures: instances

Let us now take a quick look at the (quasi)-orthogonality of sequences of column polynomials. Suppose V is a proper $*$-ideal in $\mathcal{P}_N$ and L is a Hermitian linear functional on $\mathcal{P}_N$ such that $V = \mathcal{V}_L$. Furthermore, let $\{Q_k\}_{k=0}^{\varkappa_V}$ be a rigid V-basis of $\mathcal{P}_N$ composed of real column polynomials such that $Q_0 = 1$ and $L(Q_i Q_j^*) = 0$ for all $i \neq j$. Then, by Proposition 21, the mapping

$$\mathcal{P}_N/V \times \mathcal{P}_N/V \ni (q+V, r+V) \longmapsto \langle q+V, r+V\rangle_L \stackrel{\mathrm{df}}{=} L(qr^*) \in \mathbb{C} \tag{59}$$

is a well defined sesquilinear form on $\mathcal{P}_N/V$ (because $V \subseteq \ker L$) for which there exists a rigid V-basis $\{P_k\}_{k=0}^{\varkappa_V}$ of $\mathcal{P}_N$ composed of real column polynomials such that $L(P_i P_j^*) = 0$ for all $i \neq j$ and $L(P_k P_k^*)$ is a non-singular diagonal real matrix for every $k \in \overline{0, \varkappa_V}$ (the sesquilinear form $\langle\cdot, -\rangle_L$ is a counterpart of the quasi-inner product defined in the paragraph preceding Corollary 28). Assuming moreover that $\{Q_k\}_{k=0}^{\varkappa_V}$ is L-orthonormal, we see that $\langle\cdot, -\rangle_L$ is an inner product on $\mathcal{P}_N/V$ and $\{Q_k\}_{k=0}^{\varkappa_V}$ itself can play the role of the above $\{P_k\}_{k=0}^{\varkappa_V}$ (use (43) and Theorem 36). What is more, the set $\{q+V : q \in \bigcup_{k=0}^{\varkappa_V} Q_k\}$ is an orthonormal basis of the Hilbert space completion of $\mathcal{P}_N/V$ with respect to $\langle\cdot, -\rangle_L$. This enables us to put L-orthonormality into the context of pure Hilbert space theory. Suppose further that $L = L_\mu$ with some $\mu \in \mathfrak{M}_N$. Then, by Proposition 41, $V = \mathcal{I}(\operatorname{supp}\mu) = \mathcal{I}(\mathcal{Z}_p)$ with some $p \in \mathcal{P}_N$. Hence for all $q, r \in \mathcal{P}_N$, $q = r$ a.e. $[\mu]$ if and only if $q + V = r + V$, which in turn implies that $\mathcal{P}_N/V$ can be identified with a subspace of $\mathcal{L}^2(\mu)$.

Let us turn to the assumption that $\{Q_k\}_{k=0}^{\varkappa_V}$ is a rigid V-basis of $\mathcal{P}_N$, which is L-orthonormal, and $V \subseteq \ker L$. Define the multiplication operators $M_{X_1}, \ldots, M_{X_N}$ on $\mathcal{P}_N/V$ via $M_{X_j}(q+V) = X_j q + V$ for $q \in \mathcal{P}_N$ and $j \in \overline{1, N}$. It is easily seen that $(M_{X_1}, \ldots, M_{X_N}) \in \boldsymbol{L}_{\mathrm{s}}^{\#}(\mathcal{P}_N/V)^N$ is a cyclic commuting N-tuple with the cyclic vector $X^0 + V$, where $\mathcal{P}_N/V$ is equipped with the inner product $\langle\cdot, -\rangle_L$. If the N-tuple $(M_{X_1}, \ldots, M_{X_N})$ has an extension to an N-tuple $T = (T_1, \ldots, T_N)$ of spectrally commuting self-adjoint operators acting possibly in a larger Hilbert space, then the functional L is induced by the measure $\mu(\cdot) = \langle E(\cdot)(X^0+V), X^0+V\rangle \in \mathfrak{M}_N$, where E stands for the joint spectral measure of T (see (55); the converse implication is true as well, cf. [13]). Hence μ orthonormalizes the sequence $\{Q_k\}_{k=0}^{\varkappa_V}$ and, by Theorem 36, $V = \mathcal{V}_{L_\mu}$. The following proposition sheds more light on the role played by the notion of type A in producing joint spectral measures appearing above.

Proposition 52. *Let V be a proper $*$-ideal in $\mathcal{P}_N$, $L : \mathcal{P}_N \to \mathbb{C}$ be a linear functional such that $V \subseteq \ker L$, and $\{Q_k\}_{k=0}^{\varkappa_V}$ be an L-orthonormal sequence of real column polynomials (with $Q_0 = 1$), which is a rigid V-basis of $\mathcal{P}_N$. Then for every $p \in \mathcal{P}_N$, $(p) \subseteq V$ if and only if $p(M_{X_1}, \ldots, M_{X_N}) = 0$.*

Proof. If $(p) \subseteq V$, then manifestly

$$p(M_{X_1}, \ldots, M_{X_N})(r+V) = pr + V = 0 + V \quad \text{for all } r \in \mathcal{P}_N.$$

Conversely, if $p(M_{X_1}, \dots, M_{X_N}) = 0$, then

$$p + V = p(M_{X_1}, \dots, M_{X_N})(X^0 + V) = 0 + V$$

and so the result follows. □

An obvious consequence of Proposition 52 and Theorem 36 is the following: if $p \in \mathcal{P}_N$ is of type A, then every $*$-ideal V in $\mathcal{P}_N$ containing (p) is of type C (this has already been stated in Proposition 48).

In what follows we shall focus attention on circumstances under which the three term recurrence relations modulo an ideal automatically guarantee existence of an orthonormalizing measure. Our technique allows to retrieve the well-known Favard theorem (cf. [10]).

Theorem 53. *If* $\{p_k\}_{k=0}^{\infty} \subseteq \mathcal{P}_1$ *is a sequence of real polynomials such that* $p_0 = 1$, *then the following two conditions are equivalent*:

(i) $\deg p_k = k$ *for all* $k \in \mathbb{N}$ *and there exists a measure* $\mu \in \mathfrak{M}_1$ *which orthonormalizes* $\{p_k\}_{k=0}^{\infty}$,
(ii) *for every* $k \in \mathbb{N}$, *there exist* $a_k \in \mathbb{R} \setminus \{0\}$ *and* $b_k \in \mathbb{R}$ *such that*

$$X p_k = a_k p_{k+1} + b_k p_k + a_{k-1} p_{k-1}, \quad \text{where } a_{-1} \stackrel{\text{df}}{=} 1 \quad \text{and} \quad p_{-1} \stackrel{\text{df}}{=} 0.$$

If (i) *holds, then* supp μ *is infinite*.

Proof. The equivalence (i)⇔(ii) can be deduced from Theorem 36 (with $V = \{0\}$ and $L = L_\mu$), Proposition 42 and (50). If (i) holds, then by Theorem 36, $\mathcal{V}_{L_\mu} = \{0\}$, hence by part (ii) of Proposition 39, supp μ is infinite. □

If $p \in \mathcal{P}_N$ is of type A, then Theorem 36 with $V = \mathcal{I}(\mathcal{Z}_p)$ becomes a particularly useful tool for producing measures orthonormalizing sequences of column polynomials satisfying condition (B). This can be considered as a far-reaching generalization of the Favard theorem. From this point of view, the knowledge of various classes of polynomials of type A seems to be of great importance. We now provide a brief overview of known classes of such polynomials not covered by Theorem 43. By [24, Theorem 5.4], every non-zero polynomial $p \in \mathcal{R}_2$ of degree at most 2 is of type A. An immediate adaptation of [26, Proposition 3] to our context shows that all polynomials $p \in \mathcal{P}_2$ of the form $p = X_2 + q(X_1)$ with $q \in \mathcal{P}_1$ are of type A$^{\text{o}}$. In turn, Theorem 52 of [27] asserts that $p \stackrel{\text{df}}{=} 1 + (X_1 \pm \mathrm{i} X_2) q(X_1, X_2) \in \mathcal{P}_2$ with $q \in \mathcal{P}_2$ is of type A$^{\text{o}}$. Bisgaard [6] completely characterized polynomials of the form $p = X^\alpha - X^\beta \in \mathcal{P}_N$, $\alpha, \beta \in \mathbb{N}^N$, which are of type A$^{\text{o}}$. The paper [28] contains numerous examples of polynomials p of type A$^{\text{o}}$ including

$$p = (X_1 + \mathrm{i} X_2) q(X_1, X_2) X_3 - 1 \in \mathcal{P}_3,$$

where $q \in \mathcal{P}_2$, and

$$p = (1 + q_1(X_1)^2 + \cdots + q_k(X_k)^2) r(X_1, \dots, X_k) X_{k+1} - 1 \in \mathcal{P}_{k+1},$$

where $q_1, \ldots, q_k \in \mathcal{R}_1$ are polynomials of degree at least 1 and $r \in \mathcal{R}_k$ (see [28, Remark 42] for more examples).

Despite the above instances, one may still find ideals which are not of type C.

Example 54. We indicate a proper *non-zero* $*$-ideal V in $\mathcal{P}_N$ for which there exists a rigid V-basis $\{Q_k\}_{k=0}^{\varkappa_V}$ of $\mathcal{P}_N$ (with $Q_0 = 1$) composed of real column polynomials satisfying condition (B) of Theorem 36 and such that the (positive definite) linear functional L defined by (32) is not a moment functional. Set $N = 2$ and $p = (X_2 - X_1^2)X_2 \in \mathcal{P}_2$. By [24, Theorem 6.3], the polynomial p is not of type A, and so there exists at least one positive definite linear functional L on $\mathcal{P}_2$ vanishing on (p), which is not a moment functional. Then, by Proposition 42, the ideal $V \stackrel{\text{df}}{=} \mathcal{V}_L$ is not of type C. As $\mathcal{V}_L$ is the greatest ideal contained in $\ker\ L$ and $L|_{(p)} = 0$, we conclude that

$$\{0\} \subsetneq (p) = \mathcal{I}(\mathcal{Z}_p) \subseteq \mathcal{V}_L \subseteq \ker\ L \subsetneq \mathcal{P}_2$$

(the equality $(p) = \mathcal{I}(\mathcal{Z}_p)$ is a consequence of [24, Lemma 6.1]). We do not know whether it may happen that $\mathcal{V}_L = \mathcal{I}(\mathcal{Z}_p)$ for some such L. However one can prove that for every positive definite linear functional L on $\mathcal{P}_2$, which is not a moment functional and which vanishes on (p), $\mathcal{V}_L$ is a set ideal if and only if $\mathcal{V}_L = \mathcal{I}(\mathcal{Z}_p)$. Only the "only if" part has to be justified. Suppose that $\mathcal{V}_L = \mathcal{I}(\varDelta)$ for some $\varDelta \subseteq \mathbb{R}^2$. There is no loss of generality in assuming that $\varDelta$ is closed in the Zariski topology and consequently that it is of the form $\varDelta = \mathcal{Z}_q$ with some $q \in \mathcal{P}_2$. Since $\mathcal{I}(\mathcal{Z}_p) \subseteq \mathcal{V}_L$, we get $\varDelta \subseteq \mathcal{Z}_p$. By Theorem 50 the set $\varDelta$ is infinite. Hence at least one of the two sets $\varDelta \cap \mathcal{Z}_{X_2}$ and $\varDelta \cap \mathcal{Z}_{X_2 - X_1^2}$ is infinite. Notice that the polynomials $X_2 \in \mathcal{P}_2$ and $X_2 - X_1^2 \in \mathcal{P}_2$ are both of type A (use the argument contained in the paragraph preceding Proposition 46). If $\varDelta \cap \mathcal{Z}_{X_2 - X_1^2}$ is infinite, then $q(x, x^2) = 0$ for infinitely many reals x (because $\varDelta = \mathcal{Z}_q$) and hence $q(x, x^2) = 0$ for all $x \in \mathbb{R}$, which means that $\mathcal{Z}_{X_2 - X_1^2} \subseteq \varDelta$. If the set $\varDelta' \stackrel{\text{df}}{=} \varDelta \setminus \mathcal{Z}_{X_2 - X_1^2}$ is finite, then by Theorem 50 the $*$-ideal $\mathcal{V}_L$ in $\mathcal{P}_2$ is of type C, a contradiction. If $\varDelta' \subseteq \mathcal{Z}_{X_2}$ is infinite, then we must have $\mathcal{Z}_{X_2} \subseteq \varDelta$, which in turn implies that $\mathcal{Z}_p = \mathcal{Z}_{X_2} \cup \mathcal{Z}_{X_2 - X_1^2} \subseteq \varDelta \subseteq \mathcal{Z}_p$. If $\varDelta \cap \mathcal{Z}_{X_2}$ is infinite, then the proof of the equality $\mathcal{Z}_p = \varDelta$ runs along similar lines.

The following two (surprising) questions are motivated by Example 54 (the ideal $\mathcal{V}_L$ appearing therein is not known to be a set ideal).

Question 3. Is every non-zero set ideal V in $\mathcal{P}_N$ of type C?

Question 4. Is the zero ideal in $\mathcal{P}_N$ of type C?

An answer in the negative to Question 4 implies an answer in the negative to Question 3 (with a greater N). Indeed, if $L : \mathcal{P}_N \to \mathbb{C}$ is a positive definite linear functional satisfying $\mathcal{V}_L = \{0\}$ which is not a moment functional, then the linear functional $L_1 : \mathcal{P}_{N+1} \to \mathbb{C}$ given by

$$L_1(p) = L(p(X_1, \ldots, X_N, 0)), \quad p \in \mathcal{P}_{N+1},$$

is positive definite, it is not a moment functional and $\mathcal{V}_{L_1} = (X_{N+1}) = \mathcal{I}(\Delta)$, where $\Delta = \{(x_1, \ldots, x_{N+1}) \in \mathbb{R}^{N+1} : x_{N+1} = 0\}$.

Example 55. Let (A), (B) be as in Theorem 18 and (A*), (B*) be as in Remark 27. Denote by (A**) the version of (A) in which the assumption on the rigidity of $\{Q_k\}_{k=0}^n$ is simply dropped. We show that the implication (A**)⇒(B*) (and consequently (A**)⇒(B)) is no longer true. To be more precise, we will construct a moment functional $L : \mathcal{P}_2 \to \mathbb{C}$ and a V-basis $\{Q_k\}_{k=0}^\infty$ of $\mathcal{P}_2$ composed of real polynomials satisfying (A**), (B-ii) and (B-iii), but not satisfying (A*) and (B-i).

Set $p = X_2 - X_1^2 \in \mathcal{P}_2$ and take any $\nu \in \mathfrak{M}_1$ such that $\nu(\mathbb{R}) = 1$ and $\operatorname{supp} \nu = \mathbb{R}$. Define the probability measure $\mu \in \mathfrak{M}_2$ via

$$\mu(\sigma) = \int_{\mathbb{R}} \chi_\sigma(t, t^2)\, \mathrm{d}\nu(t) \quad \text{for all Borel subsets } \sigma \text{ of } \mathbb{R}^2. \tag{60}$$

Set $L = L_\mu$ (thus L is automatically positive definite). Using (60), one can show that $\operatorname{supp} \mu = \mathcal{Z}_p$. Hence, by Proposition 39, we have

$$V \stackrel{\mathrm{df}}{=} \mathcal{I}(\mathcal{Z}_p) = \mathcal{I}(\operatorname{supp} \mu) = \mathcal{V}_L.$$

It is a matter of direct verification that $\{X_1^k\}_{k=0}^\infty$ is a V-basis of $\mathcal{P}_2$. As a consequence, $\{X_1^k\}_{k=0}^\infty$ is a basis of $F \stackrel{\mathrm{df}}{=} \operatorname{lin}\{X_1^0, X_1^1, X_1^2, \ldots\}$ ($\cong \mathcal{P}_1$), and $\mathcal{P}_2$ is the direct sum of V and F. This, (43) and $V = \mathcal{V}_L$ imply that the mapping $F \times F \ni (p, q) \longmapsto \langle p, q\rangle_L \stackrel{\mathrm{df}}{=} L(pq^*) \in \mathbb{C}$ is an inner product on F. Applying the Gram–Schmidt orthonormalization procedure to $\{X_1^0, X_1^1, X_1^2, \ldots\}$ with respect to the inner product $\langle\cdot, -\rangle_L$, we get a basis $\{Q_k\}_{k=0}^\infty$ of F composed of real polynomials such that $Q_0 = 1$ (see Remark 33), $\deg Q_i = i$ and $L(Q_i Q_j) = \delta_{i,j}$ for all $i, j \in \mathbb{N}$. This, in turn, implies that $\{Q_k\}_{k=0}^\infty$ is a V-basis of $\mathcal{P}_2$, which is L-orthonormal. Notice that $d_V(0) = 1$ and $d_V(k) = 2$ for all $k \geqslant 1$, because

$$\Sigma_0^V = X^0 \quad \text{and} \quad \Sigma_k^V = \begin{bmatrix} X_2^k \\ X_1 X_2^{k-1} \end{bmatrix} \text{ for } k \geqslant 1. \tag{61}$$

Thus $\{Q_k\}_{k=0}^\infty$ satisfies (A**), (B-ii) and (B-iii), but it does not satisfy (A*).

Since the sequence $\{Q_k\}_{k=0}^\infty$ is $L|_{\mathcal{P}_1}$-orthonormal, we infer from Theorem 53 that there exist sequences $\{a_k\}_{k=0}^\infty$ and $\{b_k\}_{k=0}^\infty$ of real numbers such that

$$X_1 Q_k = a_k Q_{k+1} + b_k Q_k + a_{k-1} Q_{k-1} \quad \text{and} \quad a_k \neq 0 \quad \text{for all } k \in \mathbb{N}, \tag{62}$$

where $a_{-1} \stackrel{\mathrm{df}}{=} 0$ and $Q_{-1} \stackrel{\mathrm{df}}{=} 0$. A direct calculation based on (62) shows that

$$\begin{aligned} X_2 Q_k \stackrel{V}{=} X_1(X_1 Q_k) = a_k a_{k+1} Q_{k+2} &+ a_k(b_k + b_{k+1}) Q_{k+1} \\ &+ (a_{k-1}^2 + a_k^2 + b_k^2) Q_k + a_{k-1}(b_{k-1} + b_k) Q_{k-1} \\ &+ a_{k-2} a_{k-1} Q_{k-2}, \quad k \in \mathbb{N}, \end{aligned} \tag{63}$$

where $a_{-2} = b_{-1} \stackrel{\mathrm{df}}{=} 0$ and $Q_{-2} \stackrel{\mathrm{df}}{=} 0$. By (62), $a_k a_{k+1} \neq 0$ for all $k \in \mathbb{N}$, and hence, by (63), $\{Q_k\}_{k=0}^\infty$ does not satisfy condition (B-i) with respect to V.

One may consider yet another column representation $\{\widetilde{Q}_k\}_{k=0}^\infty$ of the set $\{Q_0, Q_1, Q_2, \ldots\}$ given by $\widetilde{Q}_0 = Q_0$ and $\widetilde{Q}_k = [Q_{2k-1}, Q_{2k}]^\mathsf{T}$ for $k \geqslant 1$. It is obvious that $\{\widetilde{Q}_k\}_{k=0}^\infty$ is a V-basis of $\mathcal{P}_2$, which is L-orthonormal, and $\ell(\widetilde{Q}_k) = d_V(k)$ for all $k \in \mathbb{N}$. It follows from (62) and (63) that $\{\widetilde{Q}_k\}_{k=0}^{\infty}$ satisfies (B-i), (B-iii*) and (B-iv) with respect to V. Since $\deg \widetilde{Q}_k = 2k$ for $k \in \mathbb{N}$, $\{\widetilde{Q}_k\}_{k=0}^{\infty}$ is not a rigid V-basis of $\mathcal{P}_2$. However, it is possible to find a sequence $\{\widehat{Q}_k\}_{k=0}^{\infty}$ of real column polynomials satisfying (A) such that $\widehat{Q}_k \overset{V}{=} \widetilde{Q}_k$ for $k \in \mathbb{N}$. One way of obtaining this is to apply (40). The other one is to apply the explicit Gram–Schmidt orthonormalization procedure described in Remark 33 to the sequence (cf. (61))

$$X^0, \boxed{X_1, X_2}, \boxed{X_1X_2, X_2^2}, \boxed{X_1X_2^2, X_2^3}, \ldots$$

whose entries are equivalent modulo V to the entries of the following sequence:

$$X_1^0, \boxed{X_1^1, X_1^2}, \boxed{X_1^3, X_1^4}, \boxed{X_1^5, X_1^6}, \ldots .$$

12. Existence of orthogonalizing measures: matrix approach

Another way of ensuring the existence of orthonormalizing measures is to study multiplication operators $M_{X_1}, \ldots, M_{X_N}$ introduced in Section 11. Assume that V is a proper $*$-ideal in $\mathcal{P}_N$ and $\{Q_k\}_{k=0}^{\varkappa_V}$ is a sequence of real column polynomials (with $Q_0 = 1$) satisfying condition (B) of Theorem 36. By this theorem, the sequence $\{Q_k\}_{k=0}^{\varkappa_V}$ and the positive definite linear functional L defined by (32) fulfil condition (A) of Theorem 36. The space $\mathcal{P}_N/V$ is equipped with the inner product $\langle\cdot, -\rangle_L$ given by (59).

Consider first the case of $\varkappa_V < \infty$. It follows that $\mathcal{P}_N/V$ is a finite-dimensional Hilbert space. Consequently, $M_{X_1}, \ldots, M_{X_N}$ are commuting bounded self-adjoint operators and as such they admit a joint spectral measure, which yields the required orthonormalizing measure. In particular, $N = 1$ and $V \neq \{0\}$ imply $\varkappa_V < \infty$ (indeed, fixing $p \in V \setminus \{0\}$ and applying the division algorithm to each $q \in \mathcal{P}_1$, we find $r \in \mathcal{P}_1$ such that $\deg r < \deg p$ and $q - r \in (p) \subseteq V$). As $\{0\} \subseteq \mathcal{P}_1$ is of type C (cf. Theorem 53), we conclude that every $*$-ideal in $\mathcal{P}_1$ is of type C.

We now concentrate on the case $\varkappa_V = \infty$, which requires much more involved techniques. Let us arrange the set $\{q + V : q \in \bigcup_{k=0}^{\infty} Q_k\}$ in an orthonormal basis $\{q_k + V\}_{k=0}^{\infty}$ of the Hilbert space completion of $\mathcal{P}_N/V$ (relative to $\langle\cdot, -\rangle_L$) respecting the order of columns as well as the order of entries in each column. Then for every $j = 1, \ldots, N$, the multiplication operator $M_{X_j} \in \boldsymbol{L}_{\mathrm{s}}^{\#}(\mathcal{P}_N/V)$ may be represented by the infinite symmetric matrix

$$\boldsymbol{S}_j \overset{\mathrm{df}}{=} \begin{bmatrix} B_{0,j} & A_{0,j} & 0 & 0 & \\ A_{0,j}^{\top} & B_{1,j} & A_{1,j} & 0 & \ddots \\ 0 & A_{1,j}^{\top} & B_{2,j} & A_{2,j} & \ddots \\ 0 & 0 & A_{2,j}^{\top} & B_{3,j} & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

with respect to the orthonormal basis $\{q_k + V\}_{k=0}^{\infty}$, where $A_{k,j}$ and $B_{k,j}$ are the real matrices appearing in (B-i) of Theorem 36. In other words, M_{X_j} is unitarily equivalent to the matrix operator S_j defined in ℓ^2 by:

$$\mathcal{D}(S_j) = \ell_0^2, \quad S_j(a_0, a_1, a_2, \ldots) = (\boldsymbol{S}_j[a_0, a_1, a_2, \ldots]^{\mathsf{T}})^{\mathsf{T}}$$
$$\text{for } (a_0, a_1, a_2, \ldots) \in \ell_0^2,$$

where $\mathcal{D}(S_j)$ stands for the domain of S_j and ℓ_0^2 is the space of all complex sequences with finite number of non-zero entries. It is clear that the operators $M_{X_1}, \ldots, M_{X_N}$ commute. This implies that $S_1, \ldots, S_N$ commute, which is equivalent to

$$A_{k,i}A_{k+1,j} = A_{k,j}A_{k+1,i},$$
$$A_{k,i}B_{k+1,j} + B_{k,i}A_{k,j} = A_{k,j}B_{k+1,i} + B_{k,j}A_{k,i},$$
$$A_{k-1,i}^{\mathsf{T}}A_{k-1,j} + B_{k,i}B_{k,j} + A_{k,i}A_{k,j}^{\mathsf{T}} = A_{k-1,j}^{\mathsf{T}}A_{k-1,i} + B_{k,j}B_{k,i} + A_{k,j}A_{k,i}^{\mathsf{T}},$$

for all $i, j \in \overline{1, N}$ and $k \in \mathbb{N}$.

We are now in a position to formulate a criterion for the existence of orthonormalizing measures written in terms of matrices $A_{k,j}$ and $B_{k,j}$. Special cases of Theorem 56 can be found in [15,30] (in the latter the essential argument for spectral commutativity is not provided).

Theorem 56. *Let V be a proper ∗-ideal in* $\mathcal{P}_N$ *with* $\varkappa_V = \infty$ *and* $\{Q_k\}_{k=0}^{\infty}$ *be a sequence of real column polynomials* (*with* $Q_0 = 1$) *satisfying condition* (B) *of Theorem* 36. *Assume that there exists a sequence* $\{c_n\}_{n=0}^{\infty}$ *of positive real numbers such that* $\sum_{n=0}^{\infty} c_n^{-1/2} = \infty$ *and for every* $n \geqslant 0$,

$$c_n \geqslant \max\left\{\left\|\sum_{j=1}^{N} A_{n-1,j}A_{n,j}\right\|, \left\|\sum_{j=1}^{N} A_{n,j}A_{n+1,j}\right\|, \left\|\sum_{j=1}^{N}(B_{n,j}A_{n,j} + A_{n,j}B_{n+1,j})\right\|\right\}.$$

Then there exists a measure $\mu \in \mathfrak{M}_N$ *which orthonormalizes* $\{Q_k\}_{k=0}^{\infty}$ *and which satisfies the equality* $\mathcal{I}(\operatorname{supp}\mu) = V$.

Proof. We know that the matrix operators $S_1, \ldots, S_N \in \boldsymbol{L}_{\mathrm{s}}^{\#}(\ell_0^2)$ commute and $C \stackrel{\mathrm{df}}{=} S_1^2 + \cdots + S_N^2$ is a positive operator with the matrix representation

$$\boldsymbol{C} \stackrel{\mathrm{df}}{=} \begin{bmatrix} C_{0,0} & C_{0,1} & C_{0,2} & 0 & 0 & \\ C_{0,1}^{\mathsf{T}} & C_{1,1} & C_{1,2} & C_{1,3} & 0 & \ddots \\ C_{0,2}^{\mathsf{T}} & C_{1,2}^{\mathsf{T}} & C_{2,2} & C_{2,3} & C_{2,4} & \ddots \\ 0 & C_{1,3}^{\mathsf{T}} & C_{2,3}^{\mathsf{T}} & C_{3,3} & C_{3,4} & \ddots \\ 0 & 0 & C_{2,4}^{\mathsf{T}} & C_{3,4}^{\mathsf{T}} & C_{4,4} & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

where $C_{n,n} = \sum_{j=1}^{N}(A_{n-1,j}^{\mathsf{T}} A_{n-1,j} + B_{n,j}^2 + A_{n,j} A_{n,j}^{\mathsf{T}})$ with $A_{-1,j} = 0$ and

$$C_{n,n+1} = \sum_{j=1}^{N}(B_{n,j} A_{n,j} + A_{n,j} B_{n+1,j}), \quad C_{n,n+2} = \sum_{j=1}^{N} A_{n,j} A_{n+1,j}, \quad n \in \mathbb{N}. \tag{64}$$

For $n \in \mathbb{N}$, we set

$$\begin{aligned}\gamma_n(\boldsymbol{C}) &= \max\{\|C_{i,j}\| : 0 \leqslant i \leqslant n,\ j \geqslant n+1\}\\ &= \max\{\|C_{n-1,n+1}\|, \|C_{n,n+2}\|, \|C_{n,n+1}\|\} \quad (C_{-1,1} \overset{\text{df}}{=} 0).\end{aligned}$$

By our assumption and (64), $\gamma_n(\boldsymbol{C}) \leqslant c_n$ for all $n \geqslant 0$. Adapting Theorem 1 of [18] to the context of matrix operators via the discussion following Proposition 3.1 of [17] and applying it to C, we conclude that the operator $S_1^2 + \cdots + S_N^2$ is essentially self-adjoint. Hence, according to the Nelson criterion (cf. the commutative part of Theorem 5 of [21]), the closures of $S_1, \ldots, S_N$, considered as operators in ℓ^2, are spectrally commuting self-adjoint operators. This implies that the N-tuple $(M_{X_1}, \ldots, M_{X_N})$ has an extension to an N-tuple of spectrally commuting self-adjoint operators. In view of Section 11, there exists a measure $\mu \in \mathfrak{M}_N$ which orthonormalizes $\{Q_k\}_{k=0}^{\infty}$ and which satisfies the equality $\mathcal{V}_{L_\mu} = V$. Applying Proposition 39 completes the proof. $\square$

The question of (essential) spectral commutativity of matrix operators $S_1, \ldots, S_N$ can be settled in two ways: either by applying Theorem 1 of [18] and Nelson's criterion to the whole system of operators $S_1, \ldots, S_N$, like in Theorem 56, or by applying them to each pair (S_i, S_j), $1 \leqslant i < j \leqslant N$, separately, which leads to Theorem 57 below. It is not known which method is better (probably it depends on the circumstances). However, if we consider finite systems of operators with a common invariant domain, then according to Remark 28 of [28], Nelson's criterion for two operators implies that for several operators (recall that ℓ_0^2 is a common invariant domain for $S_1, \ldots, S_N$).

Theorem 57. *Let V be a proper $*$-ideal in $\mathcal{P}_N$ with $\varkappa_V = \infty$ and $\{Q_k\}_{k=0}^{\infty}$ be a sequence of real column polynomials* (*with $Q_0 = 1$*) *satisfying condition* (B) *of Theorem* 36. *Assume that for any two integers $i, j \in \overline{1, N}$ with $i < j$ there exists a sequence $\{c_n^{i,j}\}_{n=0}^{\infty}$ of positive real numbers such that $\sum_{n=0}^{\infty}(c_n^{i,j})^{-1/2} = \infty$ and $c_n^{i,j} \geqslant \max\{a_{n-1}^{i,j}, a_n^{i,j}, b_n^{i,j}\}$ for every $n \geqslant 0$, where*

$$\begin{aligned}a_n^{i,j} &\overset{\text{df}}{=} \|A_{n,i} A_{n+1,i} + A_{n,j} A_{n+1,j}\|,\\ b_n^{i,j} &\overset{\text{df}}{=} \|B_{n,i} A_{n,i} + A_{n,i} B_{n+1,i} + B_{n,j} A_{n,j} + A_{n,j} B_{n+1,j}\|.\end{aligned}$$

Then there exists a measure $\mu \in \mathfrak{M}_N$ which orthonormalizes $\{Q_k\}_{k=0}^{\infty}$ and which satisfies the equality $\mathcal{I}(\operatorname{supp} \mu) = V$.

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